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Where are the r-modes of isentropic stars?

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ABSTRACT

Almost none of the r-modes ordinarily found in rotating stars exist, if the star and its perturbations obey the same one-parameter equation of state; and rotating relativistic stars with one-parameter equations of state have no pure r-modes at all, no modes whose limit, for a star with zero angular velocity, is a perturbation with axial parity. Similarly (as we show here) rotating stars of this kind have no pure g-modes, no modes whose spherical limit is a perturbation with polar parity and vanishing perturbed pressure and density. Where have these modes gone?

In spherical stars of this kind, r-modes and g-modes form a degenerate zero-frequency subspace. We find that rotation splits the degeneracy to *zeroth* order in the star's angular velocity Ω, and the resulting modes are generically hybrids, whose limit as $\Omega \to 0$ is a stationary current with axial and polar parts. Lindblom and Ipser have recently found these hybrid modes in an analytic study of the Maclaurin spheroids. Since the hybrid modes have a rotational restoring force, they call them "rotation modes" or "generalized r-modes".

Because each mode has definite parity, its axial and polar parts have alternating values of l. We show that each mode belongs to one of two classes, axial-led or polar-led, depending on whether the spherical harmonic with lowest value of l that contributes to its velocity field is axial or polar. We numerically compute these modes for slowly rotating polytropes and for Maclaurin spheroids, using a straightforward method that appears to be novel and robust. Timescales for the gravitational-wave driven instability and for viscous damping are computed using assumptions appropriate to neutron stars. The instability to nonaxisymmetric modes is, as expected, dominated by the $l = m$ r-modes with simplest radial dependence, the only modes which retain their axial character in isentropic models; for relativistic isentropic stars, these $l = m$ modes must also be replaced by hybrids of the kind considered here.

Subject headings: instabilities — stars: neutron — stars: oscillations — stars: rotation

1. Introduction

A recently discovered instability of r-modes of rotating stars (first found in numerical study by Andersson (1998), and analytically verified by Friedman and Morsink (1998)) has gained the attention of a number of authors (Kojima 1998; Lindblom, Owen, and Morsink 1998; Owen et al. 1998; Andersson, Kokkotas and Schutz 1998; Kokkotas and Stergioulas 1998; Andersson, Kokkotas and Stergioulas 1998; Lindblom and Ipser 1998; Madsen 1998; Spruit 1999; Beyer and Kokkotas 1999; Lindblom et al. 1999). These modes have axial parity (see below) and their frequency is proportional to the star's angular velocity. Neutron stars that are rapidly rotating at birth are likely to be unstable to nonaxisymmetric perturbations driven by gravitational waves; estimates of growth times and viscous damping times (Lindblom et al 1998, Owen et al. 1998, Andersson et al 1998, Kokkotas and Stergioulas 1998, Lindblom et al. 1999) suggest that r-modes dominate the spin-down of such stars for several months, until a superfluid transition shuts off the instability. Unstable r-modes may thus set the upper limit on the spin of young neutron stars, and gravitational waves emitted during the initial spin-down might be detectable. The recent discovery by Marshall et al (1998) of a pulsar in the supernova remnant N157B implies the existence of a class of neutron stars that are rapidly rotating at birth and whose spin is plausibly limited by the gravitational-wave driven instability.

Perturbations of a spherical star can be divided into two classes, axial and polar, depending on their behavior under parity. Where polar tensor fields on a 2-sphere can be constructed from the scalars Y_l^m and their gradients ∇Y_l^m (and the metric on a 2-sphere), axial fields involve the pseudo-vector $\hat{r} \times \nabla Y_l^m$, and their behavior under parity is opposite to that of Y_l^m. That is, axial perturbations of odd l are invariant under parity, and axial perturbations with even l change sign. If a mode varies continuously along a sequence of equilibrium configurations that starts with a spherical star and continues along a path of increasing rotation, the mode will be called axial if it is axial for the spherical star. Its parity cannot change along the sequence, but l is well-defined only for modes of the spherical configuration.

It is useful to further divide stellar perturbations into subclasses according to the physics dominating their behaviour. In perfect fluid stellar models, the polar parity perturbations consist of the f-, p- and g-modes; the f- and p-modes having pressure as their dominant restoring force and the g-modes having gravity as their dominant restoring force (Cowling 1941). The axial parity perturbations are dominated by the Coriolis force in rotating stars and were called "r-modes" by Papaloizou and Pringle (1978) because of their similarity to the Rossby waves of terrestrial meteorology. For slowly rotating stars, the r-modes have frequencies which scale linearly with the star's angular velocity and in the spherical limit they become time-independent convective currents.

Despite the sudden interest in these modes, however, they are not yet well-understood for stellar models in which both the star and its perturbations are governed by a one-parameter equation of state, $p = p(\rho)$; we shall call such stellar models isentropic, because isentropic models

and their adiabatic perturbations obey the same one-parameter equation of state. For stars with more general equations of state, the r-modes appear to be complete for perturbations that have axial parity. However, this is not the case for isentropic models. One finds that the only purely axial modes allowed in isentropic stars are the physically interesting $l = m$ r-modes with simplest radial behavior (Papalouizou and Pringle 1978; Provost et al. 1978[1]; Saio 1982; Smeyers and Martens 1983). The disappearance of the purely axial modes with $l > m$ occurs for the following reason. In spherical isentropic stars the gravitational restoring forces that give rise to the g-modes vanish and they, too, become time-independent convective currents with vanishing perturbed pressure and density. Thus, the space of zero frequency modes, which generally consists only of the axial r-modes, becomes larger for spherical isentropic stars to include the polar g-modes. This large degenerate subspace of zero-frequency modes is split by rotation to zeroth order in the angular velocity, and the corresponding modes of rotating isentropic stars are hybrids whose spherical limits are mixtures of axial and polar perturbations. These hybrid modes have already been found analytically for the uniform-density Maclaurin spheroids by Lindblom and Ipser (1998) who point out that since their dominant restoring force is the Coriolis force, it is natural to refer to them as rotation modes, or generalized r-modes.

Although isentropic Newtonian stars do retain a vestigial set of purely axial modes (those having $l = m$), it appears that rotating relativistic stars of this type have *no* pure r-modes, no modes whose limit for a spherical star is purely axial (Andersson, Lockitch and Friedman, 1999). For nonisentropic relativistic stars, Kojima (1998) has derived an equation governing purely axial perturbations to lowest order in the star's angular velocity.[2] For the isentropic case, however, we find a second, independent equation for these perturbations that appears to give inconsistent radial behaviour for purely axial modes (Andersson, Lockitch and Friedman, 1999). This second equation is obtained from an angular component of the curl of the relativistic Euler equation and is simply the relativistic generalization of the equations presented in Sect. III of this paper (Eq. (36), for example). Kojima's equation is one from which polar-parity perturbations have been excluded, and, as the present paper makes clear, one cannot assume that axial and polar parity modes decouple for slowly rotating isentropic stars. Instead, we expect the Newtonian $l = m$ r-modes to become discrete axial-led hybrids of the corresponding relativistic models.

In this paper we examine the hybrid rotational modes of rotating isentropic Newtonian stars. We distinguish two types of modes, axial-led and polar-led, and show that every mode belongs to one of the two classes. We then turn to the computation of eigenfunctions and eigenfrequencies for modes in each class, adopting what appears to be a method that is both novel and robust. For

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[1]An appendix in this paper incorrectly claims that no $l = m$ r-modes exist, based on an incorrect assumption about their radial behavior

[2]Based on this equation, Kojima has argued that the spectrum is continuous, and his argument has been made precise in a recent paper of Beyer and Kokkotas (1999) (See also Kojima and Hosonuma 1999). Beyer and Kokkotas, however, also point out that the continuous spectrum they find may be an artifact of the fact that the imaginary part of the frequency vanishes in the slow-rotation limit.

the uniform-density Maclaurin spheroids, these modes have been found analytically by Lindblom and Ipser in a complementary presentation that makes certain features transparent but masks properties that are our primary concern. We examine the eigenfrequencies and corresponding eigenfunctions to lowest nontrivial order in the angular velocity Ω. We then examine the frequencies and modes of $n = 1$ polytropes, finding that the structure of the modes and their frequencies are very similar for the polytropes and the uniform-density configurations. The numerical analysis is complicated by a curious linear dependence in the Euler equations, detailed in Appendix B. The linear dependence appears in a power series expansion of the equations about the origin. It may be related to difficulty other groups have encountered in searching for these modes.

Finally, we examine unstable modes, computing their growth time and expected viscous damping time. The pure $l = m = 2$ r-mode retains its dominant role, but the $3 \leq l = m \lesssim 10$ r-modes and some of the fastest growing hybrids may contribute to the gravitational radiation and spin-down.

2. Spherical Stars

We consider a static spherically symmetric, self-gravitating perfect fluid described by a gravitational potential Φ, density ρ and pressure p. These satisfy an equation of state of the form

$$p = p(\rho), \tag{1}$$

as well as the Newtonian equilibrium equations

$$\nabla_a(h + \Phi) = 0 \tag{2}$$

$$\nabla^2\Phi = 4\pi G\rho, \tag{3}$$

where h is the specific enthalpy in a comoving frame,

$$h = \int \frac{dp}{\rho}. \tag{4}$$

We are interested in the space of zero-frequency modes, the linearized time-independent perturbations of this static equilibrium. This zero-frequency subspace is spanned by two types of perturbations: (i) perturbations with $\delta v^a \neq 0$ and $\delta\rho = \delta p = \delta\Phi = 0$, and (ii) perturbations with $\delta\rho$, δp and $\delta\Phi$ nonzero and $\delta v^a = 0$. If one assumes that no solution to the linearized equations governing a static equilibrium is spurious, that each corresponds to a family of exact solutions, then the only solutions (ii) are spherically symmetric, joining neighboring equilibria.

The decomposition into classes (i) and (ii) can be seen as follows. The set of equations satisfied by $(\delta\rho, \delta\Phi, \delta v^a)$ are the perturbed mass conservation equation,

$$\delta\left[\partial_t\rho + \nabla_a(\rho v^a)\right] = 0, \tag{5}$$

the perturbed Euler equation,

$$\delta\left[(\partial_t + \pounds_v)v_a + \nabla_a(h - \tfrac{1}{2}v^2 + \Phi)\right] = 0, \tag{6}$$

and the perturbed Poisson equation, δ[Eq. (3)].

For a time-independent perturbation these equations take the form

$$\nabla_a(\rho\delta v^a) = 0, \tag{7}$$

$$\nabla_a(\delta h + \delta\Phi) = 0, \tag{8}$$

and

$$\nabla^2\delta\Phi = 4\pi G\delta\rho, \tag{9}$$

where

$$\delta h = \frac{\delta p}{\rho} = \frac{dp}{d\rho}\frac{\delta\rho}{\rho}. \tag{10}$$

Because Eq. (7) for δv^a decouples from Eqs. (8), (9) and (10) for $(\delta\rho, \delta\Phi)$, any solution to Eqs. (7)-(10) is a superposition of a solution $(0, 0, \delta v^a)$ and a solution $(\delta\rho, \delta\Phi, 0)$. This is the claimed decomposition.

The theorem that any static self-gravitating perfect fluid is spherical implies that the solution $(\delta\rho, \delta\Phi, 0)$ is spherically symmetric, to within the assumptions that the static perturbation equations have no spurious solutions ("linearization stability")[3].

Thus, under the assumption of linearization stability we have shown that all stationary non-radial perturbations of a spherical, isentropic star have $\delta\rho = \delta p = \delta\Phi = 0$ and a velocity field δv^a that satisfies Eq. (7).

A perturbation with axial parity has the form (Friedman and Morsink 1998),

$$\delta v^a = U(r)\epsilon^{abc}\nabla_b Y_l^m \nabla_c r, \tag{11}$$

and automatically satisfies Eq. (7).

A perturbation with polar parity perturbation has the form,

$$\delta v^a = \frac{W(r)}{r}Y_l^m\nabla^a r + V(r)\nabla^a Y_l^m; \tag{12}$$

and Eq. (7) gives a relation between W and V,

$$\frac{d}{dr}(r\rho W) - l(l+1)\rho V = 0. \tag{13}$$

[3]We are aware of a proof of this linearization stability for relativistic stars under assumptions on the equation of state that would not allow polytropes (Künzle and Savage 1980).

These perturbations must satisfy the boundary conditions of regularity at the center, $r = 0$ and surface, $r = R$, of the star. Also, the Lagrangian change in the pressure (defined in the next section) must vanish at the surface of the star. These boundary conditions result in the requirement that

$$W(0) = W(R) = 0; \tag{14}$$

however, apart from this restriction, the radial functions $U(r)$ and $W(r)$ are undetermined.

Thus, a spherical, isentropic, Newtonian star admits a class of zero frequency convective fluid motions of the forms (11) and (12). Because they are stationary, these modes do not couple to gravitational radiation. [4]

3. Rotating Isentropic Stars

We consider perturbations of an isentropic Newtonian star, rotating with uniform angular velocity Ω. No assumption of slow rotation will be made until we turn to numerical computations in Sect. IV. The equilibrium of an axisymmetric, self-gravitating perfect fluid is described by the gravitational potential Φ, density ρ, pressure p and a 3-velocity

$$v^a = \Omega \varphi^a, \tag{15}$$

where φ^a is the rotational Killing vector field.

We will use a Lagrangian perturbation formalism (Friedman and Schutz 1978a) in which perturbed quantities are described in terms of a Lagrangian displacement vector ξ^a that connects fluid elements in the equilibrium and perturbed star. The Eulerian change δQ in a quantity Q is related to its Lagrangian change ΔQ by

$$\Delta Q = \delta Q + \pounds_\xi Q, \tag{16}$$

with $\pounds_\xi$ the Lie derivative along ξ^a.

The fluid perturbation is then determined by the displacement ξ^a:

$$\Delta v^a = \partial_t \xi^a \tag{17}$$

$$\frac{\Delta p}{\gamma p} = \frac{\Delta \rho}{\rho} = -\nabla_a \xi^a \tag{18}$$

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[4]Note that for spherical stars, nonlinear couplings invalidate the linear approximation after a time $t \sim R/\delta v$, comparable to the time for a fluid element to move once around the star. For nonzero angular velocity, the linear approximation is expected to be valid for all times, if the amplitude is sufficiently small, roughly, if $|\delta v| < R\Omega$.

Since the equilibrium spacetime is stationary and axisymmetric, we may decompose our perturbations into modes of the form[5] $e^{i(\sigma t+m\varphi)}$. The corresponding Eulerian changes are

$$\delta v^a = i(\sigma + m\Omega)\xi^a \tag{19}$$

$$\delta\rho = -\nabla_a(\rho\xi^a) \tag{20}$$

$$\delta p = \frac{dp}{d\rho}\delta\rho; \tag{21}$$

and the change in the gravitational potential is determined by

$$\nabla^2\delta\Phi = 4\pi G\delta\rho. \tag{22}$$

We can expand the perturbed fluid velocity, δv^a, in vector spherical harmonics (Regge and Wheeler 1957, see also Thorne 1980),

$$\delta v^a = \sum_{l=m}^{\infty}\left\{\frac{1}{r}W_l Y_l^m\nabla^a r + V_l\nabla^a Y_l^m - iU_l\epsilon^{abc}\nabla_b Y_l^m\nabla_c r\right\}e^{i\sigma t}, \tag{23}$$

and examine the perturbed Euler equation.

The Lagrangian perturbation of Euler's equation is

$$\begin{aligned} 0 &= \Delta[(\partial_t + \pounds_v)v_a + \nabla_a(h - \tfrac{1}{2}v^2 + \mathit{\Phi})] \\ &= (\partial_t + \pounds_v)\mathit{\Delta} v_a + \nabla_a[\mathit{\Delta}(h - \tfrac{1}{2}v^2 + \mathit{\Phi})], \end{aligned} \tag{24}$$

and its curl, which expresses the conservation of circulation for an isentropic star, is

$$0 = q^a \equiv i(\sigma + m\Omega)\epsilon^{abc}\nabla_b\Delta v_c, \tag{25}$$

or

$$0 = q^a = i(\sigma + m\Omega)\epsilon^{abc}\nabla_b\delta v_c + \Omega\epsilon^{abc}\nabla_b(\pounds_{\delta v}\varphi_c). \tag{26}$$

Using the spherical harmonic expansion (23) of δv^a we can write the components of q^a as

$$\begin{aligned} 0 = q^r \;=\; \frac{1}{r^2}\sum_{l=m}^{\infty}\Big\{ & [(\sigma + m\Omega)l(l+1) - 2m\Omega]U_l Y_l^m - 2\Omega V_l[\sin\theta\partial_\theta Y_l^m + l(l+1)\cos\theta Y_l^m] \\ & + 2\Omega W_l[\sin\theta\partial_\theta Y_l^m + 2\cos\theta Y_l^m]\Big\}\; e^{i\sigma t}, \end{aligned} \tag{27}$$

---

[5] We will always choose $m \geq 0$ since the complex conjugate of an $m < 0$ mode with frequency σ is an $m > 0$ mode with frequency $-\sigma$. Note that σ is the frequency in an inertial frame.

$$
\begin{aligned}
0 = q^\theta \quad = \quad & \frac{1}{r^2 \sin\theta} \sum_{l=m}^{\infty} \Big\{ m(\sigma + m\Omega) \left(\partial_r V_l - \frac{W_l}{r}\right) Y_l^m - 2\Omega \partial_r V_l \cos\theta \sin\theta \partial_\theta Y_l^m \\
& + 2\Omega m^2 \frac{V_l}{r} Y_l^m - 2\Omega \partial_r W_l \sin^2\theta Y_l^m - 2m\Omega \partial_r U_l \cos\theta Y_l^m \\
& + (\sigma + m\Omega) \partial_r U_l \sin\theta \partial_\theta Y_l^m + 2m\Omega \frac{U_l}{r} \sin\theta \partial_\theta Y_l^m \Big\} \; e^{i\sigma t},
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
0 = q^\varphi \quad = \quad & \frac{i}{r^2 \sin^2\theta} \sum_{l=m}^{\infty} \Big\{ m(\sigma + m\Omega) \partial_r U_l Y_l^m - 2\Omega \partial_r U_l \cos\theta \sin\theta \partial_\theta Y_l^m \\
& + 2\Omega \frac{U_l}{r} [m^2 - l(l+1)\sin^2\theta] Y_l^m - 2m\Omega \partial_r V_l \cos\theta Y_l^m \\
& + \left[(\sigma + m\Omega) \left(\partial_r V_l - \frac{W_l}{r}\right) + 2m\Omega \frac{V_l}{r}\right] \sin\theta \partial_\theta Y_l^m \Big\} \; e^{i\sigma t}.
\end{aligned} \tag{29}
$$

These components are not independent. The identity $\nabla_a q^a = 0$, which follows from equation (25), serves as a check on the right-hand sides of (27) - (29).

Let us rewrite these equations making use of the standard identities,

$$
\sin\theta \partial_\theta Y_l^m = l Q_{l+1} Y_{l+1}^m - (l+1) Q_l Y_{l-1}^m \tag{30}
$$

$$
\cos\theta Y_l^m = Q_{l+1} Y_{l+1}^m + Q_l Y_{l-1}^m \tag{31}
$$

where

$$
Q_l \equiv \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}. \tag{32}
$$

Defining a dimensionless comoving frequency

$$
\kappa \equiv \frac{(\sigma + m\Omega)}{\Omega}, \tag{33}
$$

we find that the $q^r = 0$ equation becomes

$$
\begin{aligned}
0 = \sum_{l=m}^{\infty} \Big\{ & [\tfrac{1}{2}\kappa l(l+1) - m] U_l Y_l^m \\
& + (W_l - l V_l)(l+2) Q_{l+1} Y_{l+1}^m - [W_l + (l+1) V_l](l-1) Q_l Y_{l-1}^m \Big\},
\end{aligned} \tag{34}
$$

$q^\theta = 0$ becomes

$$
\begin{aligned}
0 \quad = \quad & \sum_{l=m}^{\infty} \Big\{ -Q_{l+1} Q_{l+2} \left[l V_l' - W_l'\right] Y_{l+2}^m - Q_{l+1} \left[(m - \tfrac{1}{2}\kappa l) U_l' - ml \frac{U_l}{r}\right] Y_{l+1}^m \\
& + \left[\left(\tfrac{1}{2}\kappa m + (l+1) Q_l^2 - l Q_{l+1}^2\right) V_l' - \left(1 - Q_l^2 - Q_{l+1}^2\right) W_l' - \tfrac{1}{2}\kappa m \frac{W_l}{r} + m^2 \frac{V_l}{r}\right] Y_l^m
\end{aligned}
$$

$$
- Q_l \left[\left(m + \tfrac{1}{2}\kappa(l+1) \right) U_l' + m(l+1)\frac{U_l}{r} \right] Y_{l-1}^m
$$

$$
+ Q_{l-1}Q_l \left[(l+1)V_l' + W_l' \right] Y_{l-2}^m \Bigg\} \tag{35}
$$

and $q^\varphi = 0$ becomes

$$
\begin{aligned}
0 \;=\; \sum_{l=m}^{\infty} \Bigg\{ & -lQ_{l+1}Q_{l+2} \left[U_l' - (l+1)\frac{U_l}{r} \right] Y_{l+2}^m \\
& + Q_{l+1} \left[(\tfrac{1}{2}\kappa l - m)V_l' + ml\frac{V_l}{r} - \tfrac{1}{2}\kappa l \frac{W_l}{r} \right] Y_{l+1}^m \\
& + \left[\left(\tfrac{1}{2}\kappa m + (l+1)Q_l^2 - lQ_{l+1}^2 \right) U_l' + \left(m^2 - l(l+1)\left(1 - Q_l^2 - Q_{l+1}^2\right) \right) \frac{U_l}{r} \right] Y_l^m \\
& - Q_l \left[\left(\tfrac{1}{2}\kappa(l+1) + m \right) V_l' + m(l+1)\frac{V_l}{r} - \tfrac{1}{2}\kappa(l+1)\frac{W_l}{r} \right] Y_{l-1}^m \\
& + (l+1)Q_{l-1}Q_l \left[U_l' + l\frac{U_l}{r} \right] Y_{l-2}^m \Bigg\}
\end{aligned} \tag{36}
$$

where $' \equiv \frac{d}{dr}$.

From this last form of the equations it is clear that the rotation of the star mixes the axial and polar contributions to δv^a. That is, rotation mixes those terms in (23) whose limit as $\Omega \to 0$ is axial with those terms in (23) whose limit as $\Omega \to 0$ is polar. It is also evident that the axial contributions to δv^a with l even mix only with the odd l polar contributions, and that the axial contributions with l odd mix only with the even l polar contributions. In addition, we prove in appendix A that for non-axisymmetric modes the lowest value of l that appears in the expansion of δv^a is always $l = m$ (When $m = 0$ this lowest value of l is either 0 or 1.)

Thus, we find two distinct classes of mixed, or hybrid, modes with definite behavior under parity. This is to be expected because a rotating star is invariant under parity. Let us call a non-axisymmetric[6] mode an "axial-led hybrid" (or simply "axial-hybrid") if δv^a receives contributions only from

$$
\begin{aligned}
\text{axial terms with } l &= m,\ m+2,\ m+4,\ \ldots \text{ and} \\
\text{polar terms with } l &= m+1,\ m+3,\ m+5,\ \ldots.
\end{aligned}
$$

Such a mode has parity $(-1)^{m+1}$.

Similarly, we define a non-axisymmetric[7] mode to be a "polar-led hybrid" (or "polar-hybrid")

[6]When $m = 0$ there exists a set of modes with parity $+1$ that may be designated as "axial-led hybrids" since δv^a receives contributions only from axial terms with $l = 1, 3, 5, \ldots$ and polar terms with $l = 2, 4, 6, \ldots$.

[7]When $m = 0$ there exist two sets of modes that may be designated as "polar-led hybrids." One set has parity -1 and δv^a receives contributions only from polar terms with $l = 1, 3, 5, \ldots$ and axial terms with $l = 2, 4, 6, \ldots$. The other set (which includes the radial oscillations) has parity $+1$ and δv^a receives contributions only from polar terms with $l = 0, 2, 4, \ldots$ and axial terms with $l = 1, 3, 5, \ldots$.

if δv^a receives contributions only from

$$\begin{aligned} \text{polar terms with } l &= m,\ m+2,\ m+4,\ \ldots \text{ and} \\ \text{axial terms with } l &= m+1,\ m+3,\ m+5,\ \ldots. \end{aligned}$$

Such a mode has parity $(-1)^m$.

Let us rewrite the equations one last time using the orthogonality relation for spherical harmonics,

$$\int Y_{l'}^{m'} Y_l^{*m} d\Omega = \delta_{ll'}\delta_{mm'}, \tag{37}$$

where $d\Omega$ is the usual solid angle element.

From equation (34) we find that $\int q^r Y_l^{*m} d\Omega = 0$ gives

$$0 = [\tfrac{1}{2}\kappa l(l+1) - m]U_l + (l+1)Q_l[W_{l-1} - (l-1)V_{l-1}] - lQ_{l+1}[W_{l+1} + (l+2)V_{l+1}] \tag{38}$$

Similarly, $\int q^\theta Y_l^{*m} d\Omega = 0$ gives

$$\begin{aligned} 0 \;=\;& Q_l Q_{l-1}\{(l-2)V'_{l-2} - W'_{l-2}\} + Q_l\left\{[m - \tfrac{1}{2}\kappa(l-1)]U'_{l-1} - m(l-1)\frac{U_{l-1}}{r}\right\} \\ &+ \left(1 - Q_l^2 - Q_{l+1}^2\right)W'_l - \left[\tfrac{1}{2}\kappa m + (l+1)Q_l^2 - lQ_{l+1}^2\right]V'_l + \tfrac{1}{2}\kappa m\frac{W_l}{r} - m^2\frac{V_l}{r} \\ &+ Q_{l+1}\left\{[m + \tfrac{1}{2}\kappa(l+2)]U'_{l+1} + m(l+2)\frac{U_{l+1}}{r}\right\} \\ &- Q_{l+2}Q_{l+1}\{(l+3)V'_{l+2} + W'_{l+2}\} \end{aligned} \tag{39}$$

and $\int q^\varphi Y_l^{*m} d\Omega = 0$ gives

$$\begin{aligned} 0 \;=\;& -(l-2)Q_lQ_{l-1}\left[U'_{l-2} - (l-1)\frac{U_{l-2}}{r}\right] + (l+3)Q_{l+2}Q_{l+1}\left[U'_{l+2} + (l+2)\frac{U_{l+2}}{r}\right] \\ &+ \left\{\left[\tfrac{1}{2}\kappa m + (l+1)Q_l^2 - lQ_{l+1}^2\right]U'_l + \left[m^2 - l(l+1)\left(1 - Q_l^2 - Q_{l+1}^2\right)\right]\frac{U_l}{r}\right\} \\ &+ Q_l\left\{[\tfrac{1}{2}\kappa(l-1) - m]V'_{l-1} + m(l-1)\frac{V_{l-1}}{r} - \tfrac{1}{2}\kappa(l-1)\frac{W_{l-1}}{r}\right\} \\ &- Q_{l+1}\left\{[\tfrac{1}{2}\kappa(l+2) + m]V'_{l+1} + m(l+2)\frac{V_{l+1}}{r} - \tfrac{1}{2}\kappa(l+2)\frac{W_{l+1}}{r}\right\}. \end{aligned} \tag{40}$$

4. Method of Solution

In our numerical solution, we restrict consideration to slowly rotating stars, finding axial- and polar-led hybrids to lowest order in the angular velocity Ω. That is, we assume that perturbed quantities introduced above obey the following ordering in Ω:

$$\begin{aligned} &W_l \sim O(1), \quad V_l \sim O(1), \quad U_l \sim O(1), \\ &\delta\rho \sim O(\Omega), \quad \delta p \sim O(\Omega), \quad \delta\Phi \sim O(\Omega), \quad \sigma \sim O(\Omega). \end{aligned} \tag{41}$$

The $\Omega \to 0$ limit of such a perturbation is a sum of the zero-frequency axial and polar perturbations considered in Sect. II. Note that, although the relative orders of $\delta\rho$ and δv^a are physically meaningful, there is an arbitrariness in their absolute order. If $(\delta\rho, \delta v^a)$ is a solution to the linearized equations, so is $(\Omega\delta\rho, \Omega\delta v^a)$. We have chosen the order (41) to reflect the existence of well-defined, nontrivial velocity perturbations of the spherical model. Other authors (e.g., Lindblom and Ipser (1998)) adopt a convention in which $\delta v^a = O(\Omega)$ and $\delta\rho = O(\Omega^2)$.

To lowest order, the equations governing these perturbations are the perturbed Euler equations (38) - (40) and the perturbed mass conservation equation, (7), which becomes

$$rW_l' + \left(1 + r\frac{\rho'}{\rho}\right) W_l - l(l+1)V_l = 0. \tag{42}$$

In addition, the perturbations must satisfy the boundary conditions of regularity at the center of the star, $r = 0$, regularity at the surface of the star, $r = R$, and the vanishing of the Lagrangian change in the pressure at the surface of the star,

$$0 = \Delta p \equiv \delta p + \pounds_\xi p = \xi^r p' + \mathrm{O}(\Omega). \tag{43}$$

Equations (38) through (42) are a system of ordinary differential equations for $W_{l'}(r)$, $V_{l'}(r)$ and $U_{l'}(r)$ (for all l'). Together with the boundary conditions, these equations form a non-linear eigenvalue problem for the parameter κ, where $\kappa\Omega$ is the mode frequency in the rotating frame.

To solve for the eigenvalues we proceed as follows. We first ensure that the boundary conditions are automatically satisfied by expanding $W_{l'}(r)$, $V_{l'}(r)$ and $U_{l'}(r)$ (for all l') in regular power series about the surface and center of the star. Substituting these series into the differential equations results in a set of algebraic equations for the expansion coefficients. These algebraic equations may be solved for arbitrary values of κ using standard matrix inversion methods. For arbitrary values of κ, however, the series solutions about the center of the star will not necessarily agree with those about the surface of the star. The requirement that the series agree at some matching point, $0 < r_0 < R$, then becomes the condition that restricts the possible values of the eigenvalue, κ_0.

The equilibrium solution (ρ, Φ) appears in the perturbation equations only through the quantity (ρ'/ρ) in equation (42). We begin by writing the series expansion for this quantity about $r = 0$ as

$$\left(\frac{\rho'}{\rho}\right) = \frac{1}{R} \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} \pi_i \left(\frac{r}{R}\right)^i, \tag{44}$$

and about $r = R$ as

$$\left(\frac{\rho'}{\rho}\right) = \frac{1}{R} \sum_{k=-1}^{\infty} \tilde{\pi}_k \left(1 - \frac{r}{R}\right)^k, \tag{45}$$

where the π_i and $\tilde{\pi}_k$ are determined from the equilibrium solution.

Because (38) relates $U_l(r)$ algebraically to $W_{l\pm1}(r)$ and $V_{l\pm1}(r)$, we may eliminate $U_{l'}(r)$ (all l') from (39) and (40). We then need only work with one of equations (39) or (40) since the equations (38) through (40) are related by $\nabla_a q^a = 0$.

We next replace ρ'/ρ, $W_{l'}$, and $V_{l'}$ in equations (39) or (40) by their series expansions. We eliminate the $U_{l'}(r)$ from either (39) or (40) and, again, substitute for the $W_{l'}(r)$ and $V_{l'}(r)$. Finally, we write down the matching condition at the point r_0 equating the series expansions about $r = 0$ to the series expansions about $r = R$. (Explicitly one equates (B6) and (B7) of Appendix B for axial-led modes or (B13) and (B14) for polar-led modes). The result is a linear algebraic system which we may represent schematically as

$$Ax = 0. \tag{46}$$

In this equation, A is a matrix which depends non-linearly on the parameter κ, and x is a vector whose components are the unknown coefficients in the series expansions for the $W_{l'}(r)$ and $V_{l'}(r)$. In Appendix B, we explicitly present the equations making up this algebraic system as well as the forms of the regular series expansions for $W_{l'}(r)$ and $V_{l'}(r)$.

To satisfy equation (46) we must find those values of κ for which the matrix A is singular, i.e., we must find the zeroes of the determinant of A. We truncate the spherical harmonic expansion of δv^a at some maximum index $l_{\max}$ and we truncate the radial series expansions about $r = 0$ and $r = R$ at some maximum powers $i_{\max}$ and $k_{\max}$, respectively.

The resulting finite matrix is band diagonal. To find the zeroes of its determinant we use standard root finding techniques combined with routines from the LAPACK linear algebra libraries (Anderson et al. 1994). We find that the eigenvalues, κ_0, computed in this manner converge quickly as we increase $l_{\max}$, $i_{\max}$ and $k_{\max}$.

The eigenfunctions associated with these eigenvalues are determined by the perturbation equations only up to normalization. Given a particular eigenvalue, we find its eigenfunction by replacing one of the equations in the system (46) with the normalization condition that

$$\begin{array}{ll} V_m(r = R) = 1 & \text{for polar-hybrids, or that} \\ V_{m+1}(r = R) = 1 & \text{for axial-hybrids.} \end{array} \tag{47}$$

Since we have eliminated one of the rows of the singular matrix A in favor of this condition, the result is an algebraic system of the form

$$\tilde{A}x = b, \tag{48}$$

where $\tilde{A}$ is now a non-singular matrix and b is a known column vector. We solve this system for the vector x using routines from LAPACK and reconstruct the various series expansions from this solution vector of coefficients.

5. The Eigenvalues and Eigenfunctions

We have computed the eigenvalues and eigenfunctions for uniform density stars and for $n = 1$ polytropes, models obeying the polytropic equation of state $p = K\rho^2$, where K is a constant. Our numerical solutions for the uniform density star agree with the recent results of Lindblom and Ipser (1998) who find analytic solutions for the hybrid modes in rigidly rotating uniform density stars with arbitrary angular velocity - the Maclaurin spheroids. Their calculation uses the two-potential formalism (Ipser and Managan 1985; and Ipser and Lindblom 1990) in which the equations for the perturbation modes are reformulated as coupled differential equations for a fluid potential, δU, and the gravitational potential, $\delta\Phi$. All of the perturbed fluid variables may be expressed in terms of these two potentials. The analysis follows that of Bryan (1889) who found that the equations are separable in a non-standard spheroidal coordinate system.

The Bryan/Lindblom-Ipser eigenfunctions δU_0 and $\delta\Phi_0$ turn out to be products of associated Legendre polynomials of their coordinates. This simple form of their solutions leads us to expect that our series solutions might also have a simple form - even though their unusual spheroidal coordinates are rather complicated functions of r and θ. In fact, we do find that the modes of the uniform density star have a particularly simple structure. For any particular mode, both the angular and radial series expansions terminate at some finite indices l_0 and i_0 (or k_0). That is, the spherical harmonic expansion (23) of δv^a contains only terms with $m \leq l \leq l_0$ for this mode, and the coefficients of this expansion - the $W_l(r)$, $V_l(r)$ and $U_l(r)$ - are polynomials of order i_0. For all $l_0 \geq m$ there exist a number of modes terminating at l_0.

In Tables 1 to 4 we present the functions $W_l(r)$, $V_l(r)$ and $U_l(r)$ for all of the axial- and polar-led hybrids with $m = 1$ and $m = 2$ for a range of values of the terminating index l_0. (See also Figure 1.) For given values of $m > 0$ and l_0 there are $l_0 - m + 1$ modes. (When $m = 0$ there are l_0 modes. See equation (50) below.) We also find that the last term in the expansion (23), the term with $l = l_0$, is always axial for both types of hybrid modes. This fact, together with the fact that the parity of the modes is,

$$\pi = \begin{cases} (-1)^m & \text{for polar-led hybrids} \\ (-1)^{m+1} & \text{for axial-led hybrids,} \end{cases} \tag{49}$$

(for $m > 0$) implies that $l_0 - m + 1$ must be even for polar-led modes and odd for axial-led modes.

The fact that the various series terminate at l_0, i_0 and k_0 implies that Equations (46) and (48) will be exact as long as we truncate the series at $l_{\max} \geq l_0$, $i_{\max} \geq i_0$ and $k_{\max} \geq k_0$.

To find the eigenvalues of these modes we search the κ axis for all of the zeroes of the determinant of the matrix A in equation (46). We begin by fixing m and performing the search with $l_{\max} = m$. We then increase $l_{\max}$ by 1 and repeat the search (and so on). At any given value of $l_{\max}$, the search finds all of the eigenvalues associated with the eigenfunctions terminating at $l_0 \leq l_{\max}$.

In Table 5, we present the eigenvalues κ_0 found by this method for the axial- and polar-led

hybrid modes of uniform density stars for a range of values of l_0 and m. Observe that many of the eigenvalues, (marked with a $*$) satisfy the condition $\sigma(\sigma + m\Omega) < 0$. [Recall that the mode frequency in an inertial frame is $\sigma = (\kappa_0 - m)\Omega$]. The modes whose frequencies satisfy this condition are subject to a gravitational radiation driven instability in the absence of viscosity. The modes having $l_0 = m > 0$ (or $l_0 = 1$ for $m = 0$) are the purely axial r-modes. Their frequencies were found by Papalouizou and Pringle (1978) and are given by $\kappa_0 = 2/(m+1)$ (or $\kappa_0 = 0$ for $m = 0$). We find that there are no purely polar modes satisfying our assumptions (41) in these stellar models.

We have compared these eigenvalues with those of Lindblom and Ipser (1998). To lowest non-trivial order in Ω their equation for the eigenvalue, κ_0, can be expressed in terms of associated Legendre polynomials[8] (see Lindblom and Ipser's equation 6.4), as

$$(4 - \kappa_0^2)\frac{d}{d\kappa}P^m_{l_0+1}(\tfrac{\kappa_0}{2}) - 2mP^m_{l_0+1}(\tfrac{\kappa_0}{2}) = 0. \tag{50}$$

For given values of $m > 0$ and l_0 this equation has $l_0 - m + 1$ roots (corresponding to the number of distinct modes), which can easily be found numerically. (For $m = 0$ there are l_0 roots.) For the range of values of m and l_0 checked our eigenvalues agree with these to machine precision. (Compare our Table 5 with Table 1 in Lindblom and Ipser 1998.)

We have also compared our eigenfunctions with those of Lindblom and Ipser. For a uniformly rotating, isentropic star, the fluid velocity perturbation, δv^a, is related (Ipser and Lindblom 1990) to δU by

$$\nabla_a \delta U = -\left[i\kappa\Omega g_{ab} + 2\nabla_b v_a\right]\delta v^b. \tag{51}$$

Since the φ component of this equation is simply

$$im\delta U = -\Omega r^2 \sin^2\theta \left[\frac{2}{r}\delta v^r + 2\cot\theta\delta v^\theta + i\kappa\delta v^\varphi\right], \tag{52}$$

it is straightforward numerically to construct this quantity from the components of our δv^a and compare it with the analytic solutions for δU given by Lindblom and Ipser (see their equation 7.2). We have compared these solutions on a 20×40 grid in the $(r - \theta)$ plane and found that they agree (up to normalization) to better than 1 part in 10^9 for all cases checked.

Because of the use of the two-potential formalism and the unusual coordinate system used in their analysis, the axial or polar hybrid character of the Bryan/Lindblom-Ipser solutions is not obvious. Nor is it evident that these solutions have, as their $\Omega \to 0$ limit, the zero-frequency convective modes described in Sect. II. The comparison of their analytic results with our numerical work has served the dual purpose of clarifying these properties of the solutions and of testing the accuracy of our code. The computational differences are minor between the uniform density

[8]The index l used by Lindblom and Ipser is related to our l_0 by $l = l_0 + 1$. Our convention agrees with the usual labelling of the $l_0 = m$ pure axial modes.

calculation and one in which the star obeys a more realistic equation of state. Thus, this testing gives us confidence in the validity of our code for the polytrope calculation. As a further check, we have written two independent codes and compared the eigenvalues computed from each. One of these codes is based on the set of equations described in Appendix B. The other is based on the set of second order equations that results from using the mass conservation equation, (42), to substitute for all the $V_l(r)$ in favor of the $W_l(r)$.

For the $n = 1$ polytrope we will consider and, more generally, for any isentropic equation of state, the purely axial r-modes are independent of the equation of state. In both isentropic and non-isentropic stars, pure r-modes exist whose velocity field is, to lowest order in Ω, an axial vector field belonging to a single angular harmonic (and restricted to harmonics with $l = m$ in the isentropic case). The frequency of such a mode is given (to order Ω) by $\kappa\Omega = (\sigma + m\Omega) = 2m\Omega/l(l+1)$ (Papalouizou and Pringle 1978) and is independent of the equation of state. In isentropic stars, only those modes having $l = m$ (or $l = 1$ for $m = 0$) exist, and for these modes the eigenfunctions are also independent of the (isentropic) equation of state[9]. This independence of the equation of state occurs for the r-modes because (to lowest order in Ω) fluid elements move in surfaces of constant r (and thus in surfaces of constant density and pressure). For the hybrid modes, however, fluid elements are not confined to surfaces of constant r and one would expect the eigenfrequencies and eigenfunctions to depend on the equation of state.

Indeed, we find such a dependence. The hybrid modes of the $n = 1$ polytrope are not identical to those of the uniform density star. On the other hand, the modes do not appear to be very sensitive to the equation of state. We have found that the character of the polytropic modes is similar to the modes of the uniform density star, except that the radial and angular series expansions do not terminate. For each eigenfunction in the uniform density star there is a corresponding eigenfunction in the polytrope with a slightly different eigenfrequency (See Table 6.) For a given mode of the uniform density star, the series expansion (23) terminates at $l = l_0$. For the corresponding polytrope mode, the expansion (23) does not terminate, but it does converge quickly. The largest terms in (23) with $l > l_0$ are more than an order of magnitude smaller than those with $l \leq l_0$ and they decrease rapidly as l increases. Thus, the terms that dominate the polytrope eigenfunctions are those that correspond to the non-zero terms in the corresponding uniform density eigenfunctions.

In Figures 1 and 2 we display the coefficients $W_l(r)$, $V_l(r)$ and $U_l(r)$ of the expansion (23) for the same $m = 2$ axial-led hybrid mode in each stellar model. For the uniform density star (Figure 1) the only non-zero coefficients for this mode are those with $l \leq l_0 = 4$. These coefficients are presented explicitly in Table 3 and are low order polynomials in r. For the corresponding mode in the polytrope, we present in Figure 2 the first seven coefficients of the expansion (23). Observe

[9]The only non-zero term in (23) for these modes is the axial $l_0 = m$ term with coefficient $U_m(r) = r^{m+1}$ (Provost et al. 1981). The purely axial mode with $m = 0$ has $l_0 = 1$ and radial dependence r^2. This mode has zero frequency and corresponds to a small uniform change in the angular velocity of the star.

that those coefficients with $l \leq 4$ are similar to the corresponding functions in the uniform density mode and dominate the polytrope eigenfunction. The coefficients with $4 < l \leq 6$ are an order of magnitude smaller than the dominant coefficients and those with $l > 6$ are smaller still. (Since they would be indistinguishable from the (r/R) axis, we do not display the coefficients having $l > 6$ for this mode.)

Just as the angular series expansion fails to terminate for the polytrope modes, so too do the radial series expansions for the functions $W_l(r)$, $V_l(r)$ and $U_l(r)$. We have seen that in the uniform density star these functions are polynomials in r (Tables 1 through 4). In the polytropic star, the radial series do not terminate and we are required to work with both sets of radial series expansions - those about the center of the star and those about its surface - in order to represent the functions accurately everywhere inside the star.

In Figures 3 through 11 we compare corresponding functions from each type of star. For example, Figures 3, 4, and 5 show the functions $W_l(r)$, $V_l(r)$ and $U_l(r)$ (respectively) for $l \leq 6$ for a particular $m = 1$ polar-led hybrid mode. In the uniform density star this mode has eigenvalue $\kappa_0 = 1.509941$, and in the polytrope it has eigenvalue $\kappa_0 = 1.412999$. The only non-zero functions in the uniform density mode are those with $l \leq l_0 = 2$ and they are simple polynomials in r (see Table 2). Observe that these functions are similar, but not identical to, their counterparts in the polytrope mode, which have been constructed from their radial series expansions about $r = 0$ and $r = R$ (with matching point $r_0 = 0.5R$). Again, note the convergence with increasing l of the polytrope eigenfunction. The mode is dominated by the terms with $l \leq 2$ and those with $l > 2$ decrease rapidly with l. (The $l = 5$ and $l = 6$ coefficients are virtually indistinguishable from the (r/R) axis.)

Because the polytrope eigenfunctions are dominated by their $l \leq l_0$ terms, the eigenvalue search with $l_{\rm max} = l_0$ will find the associated eigenvalues approximately. We compute these approximate eigenvalues of the polytrope modes using the same search technique as for the uniform density star. We then increase $l_{\rm max}$ and search near one of the approximate eigenvalues for a corrected value, iterating this procedure until the eigenvalue converges to the desired accuracy. We present the eigenvalues found by this method in Table 6.

As a further comparison between the mode eigenvalues in the polytropic star and those in the uniform density star we have modelled a sequence of "intermediate" stars. By multiplying the expansions (44) and (45) for (ρ'/ρ) by a scaling factor, $\epsilon \in [0, 1]$, we can simulate a continuous sequence of stellar models connecting the uniform density star ($\epsilon = 0$) to the polytrope ($\epsilon = 1$). We find that an eigenvalue in the uniform density star varies smoothly as function of ϵ to the corresponding eigenvalue in the polytrope.

6. The Effects of Dissipation

The effects of gravitational radiation and viscosity on the pure $l_0 = m$ r-modes have already been studied by a number of authors. (Lindblom et al 1998, Owen et al. 1998, Andersson et al 1998, Kokkotas and Stergioulas 1998, Lindblom et al. 1999) All of these modes are unstable to gravational radiation reaction, and for some of them this instability strongly dominates viscous damping. We now consider the effects of dissipation on the axial- and polar-hybrid modes.

To estimate the timescales associated with viscous damping and gravitational radiation reaction we follow the methods used for the $l_0 = m$ modes (Lindblom et al 1998, see also Ipser and Lindblom 1991). When the energy radiated per cycle is small compared to the energy of the mode, the imaginary part of the mode frequency is accurately approximated by the expression

$$\frac{1}{\tau} = -\frac{1}{2E}\frac{dE}{dt}, \tag{53}$$

where E is the energy of the mode as measured in the rotating frame,

$$E = \frac{1}{2}\int \left[\rho\delta v^a \delta v_a^* + \left(\frac{\delta p}{\rho} + \delta\Phi\right)\delta\rho^*\right] d^3x. \tag{54}$$

The rate of change of this energy due to dissipation by viscosity and gravitational radiation is,

$$\begin{aligned}\frac{dE}{dt} &= -\int \left(2\eta\delta\sigma^{ab}\delta\sigma^*_{ab} + \zeta\delta\theta\delta\theta^*\right) \\ &\quad -\sigma(\sigma+m\Omega)\sum_{l\geq 2} N_l \sigma^{2l}\left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right).\end{aligned} \tag{55}$$

The first term in (55) represents dissipation due to shear viscosity, where the shear, $\delta\sigma_{ab}$, of the perturbation is

$$\delta\sigma_{ab} = \tfrac{1}{2}\left(\nabla_a \delta v_b + \nabla_b \delta v_a - \tfrac{2}{3} g_{ab}\nabla_c \delta v^c\right), \tag{56}$$

and the coefficient of shear viscosity for hot neutron-star matter is (Cutler and Lindblom 1987; Sawyer 1989)

$$\eta = 2\times 10^{18}\left(\frac{\rho}{10^{15}\mathrm{g\cdot cm^{-3}}}\right)^{\frac{9}{4}}\left(\frac{10^9 K}{T}\right)^2 \ \mathrm{g\cdot cm^{-1}\cdot s^{-1}}. \tag{57}$$

The second term in (55) represents dissipation due to bulk viscosity, where the expansion, $\delta\theta$, of the perturbation is

$$\delta\theta = \nabla_c \delta v^c \tag{58}$$

and the bulk viscosity coefficient for hot neutron star matter is (Cutler and Lindblom 1987; Sawyer 1989)

$$\zeta = 6\times 10^{25}\left(\frac{1\mathrm{Hz}}{\sigma+m\Omega}\right)^2\left(\frac{\rho}{10^{15}\mathrm{g\cdot cm^{-3}}}\right)^2\left(\frac{T}{10^9 K}\right)^6 \ \mathrm{g\cdot cm^{-1}\cdot s^{-1}}. \tag{59}$$

The third term in (55) represents dissipation due to gravitational radiation, with coupling constant

$$N_l = \frac{4\pi G}{c^{2l+1}} \frac{(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}. \tag{60}$$

The mass, δD_{lm}, and current, δJ_{lm}, multipole moments of the perturbation are given by (Thorne 1980, Lindblom et al 1998)

$$\delta D_{lm} = \int \delta\rho r^l Y_l^{*m} d^3x, \tag{61}$$

and

$$\delta J_{lm} = \frac{2}{c}\left(\frac{l}{l+1}\right)^{\frac{1}{2}} \int r^l \left(\rho\delta v_a + \delta\rho v_a\right) Y_{lm}^{a,B*} d^3x \tag{62}$$

where $Y_{lm}^{a,B}$ is the magnetic type vector spherical harmonic (Thorne 1980) given by,

$$Y_{lm}^{a,B} = \frac{r}{\sqrt{l(l+1)}} \epsilon^{abc} \nabla_b Y_l^m \nabla_c r. \tag{63}$$

To lowest order in Ω, the energy (54) of the hybrid modes is positive definite. Their stability is therefore determined by the sign of the right hand side of equation (55). We have seen that many of the hybrid modes have frequencies satisfying $\sigma(\sigma + m\Omega) < 0$. This makes the third term in (55) positive, implying that gravitational radiation reaction tends to drive these modes unstable. (Chandrasekhar 1970; Friedman and Schutz 1978b; Friedman 1978) To determine the actual stability of these modes, we must evaluate the various dissipative terms in (55).

We first substitute for δv^a the spherical harmonic expansion (23) and use the orthogonality relations for vector spherical harmonics (Thorne 1980) to perform the angular integrals. The energy of the modes in the rotating frame then becomes

$$E = \sum_{l=m}^{\infty} \tfrac{1}{2} \int_0^R \rho \left[W_l^2 + l(l+1)V_l^2 + l(l+1)U_l^2\right] dr. \tag{64}$$

To calculate the dissipation due to gravitational radiation reaction we must evaluate the multipole moments (61) and (62). To lowest order in Ω the mass multipole moments vanish and the current multipole moments are given by

$$\delta J_{lm} = \frac{2l}{c} \int_0^R \rho r^{l+1} U_l dr. \tag{65}$$

To calculate the dissipation due to bulk viscosity we must evaluate the expansion, $\delta\theta = \nabla_c \delta v^c$, of the perturbation. For uniform density stars this quantity vanishes identically by the mass conservation equation (7). For the $l_0 = m$, pure axial modes the expansion, again, vanishes identically, regardless of the equation of state. To compute the bulk viscosity of these modes it is necessary to work to higher order in Ω (Andersson et al. 1998, Lindblom et al. 1999). On the other hand, for the new hybrid modes in which we are interested, the expansion of the fluid perturbation

is non-zero in the slowly rotating polytropic stars. After substituting for δv^a its series expansion and performing the angular integrals, the bulk viscosity contribution to (55) becomes

$$\left(\frac{dE}{dt}\right)_B = -\sum_{l=m}^{\infty}\int_0^R \frac{\zeta}{r^2}\left[rW_l' + W_l - l(l+1)V_l\right]^2 dr \tag{66}$$

In a similar manner, the contribution to (55) from shear viscosity becomes

$$\begin{aligned}\left(\frac{dE}{dt}\right)_S = -\sum_{l=m}^{\infty}\int_0^R \frac{2\eta}{r^2}\, &\left\{\tfrac{2}{3}\left[r^3\left(\tfrac{W_l}{r^2}\right)'\right]^2 + \tfrac{1}{2}l(l+1)W_l^2 + \tfrac{1}{2}l(l+1)\left[r^3\left(\tfrac{V_l}{r^2}\right)'\right]^2\right.\\ &+\tfrac{1}{3}l(l+1)(2l^2+2l-3)V_l^2 + l(l+1)W_l\left[r^5\left(\tfrac{V_l}{r^4}\right)'\right] + \tfrac{2}{3}l(l+1)V_l\,(rW_l)'\\ &\left.+\tfrac{1}{2}l(l+1)\left[r^3\left(\tfrac{U_l}{r^2}\right)'\right]^2 + \tfrac{1}{2}l(l+1)(l^2+l-2)U_l^2\right\}dr.\end{aligned} \tag{67}$$

Given a numerical solution for one of the hybrid mode eigenfunctions, these radial integrals can be performed numerically. The resulting contributions to (55) also depend on the angular velocity and temperature of the star. Let us express the imaginary part of the hybrid mode frequency (53) as,

$$\frac{1}{\tau} = \frac{1}{\tilde{\tau}_S}\left(\frac{10^9K}{T}\right)^2 + \frac{1}{\tilde{\tau}_B}\left(\frac{T}{10^9K}\right)^6\left(\frac{\pi G\bar{\rho}}{\Omega^2}\right) + \sum_{l\geq 2}\frac{1}{\tilde{\tau}_l}\left(\frac{\Omega^2}{\pi G\bar{\rho}}\right)^{l+1}, \tag{68}$$

where $\bar{\rho}$ is average density of the star. (Compare this expression to the corresponding expression in Lindblom et al. (1998) - their equation (22) - for the $l_0 = m$ pure axial modes.)

The bulk viscosity term in this equation is stronger by a factor Ω^{-4} than that for the $l_0 = m$ pure axial modes. This is because the expansion $\delta\theta$ of the hybrid mode is nonzero to lowest order in Ω for the polytropic star, whereas it is order Ω^2 for the pure axial modes. This implies that the damping due to bulk viscosity will be much stronger for the hybrid modes than for the pure axial modes in slowly rotating stars.

Note that the contribution to (68) from gravitational radiation reaction consists of a sum over all the values of l with a non-vanishing current multipole. This sum is, of course, dominated by the lowest contributing multipole.

In Tables 7 to 9 we present the timescales for these various dissipative effects in the uniform density and polytropic stellar models that we have been considering with $R = 12.57$km and $M = 1.4M_\odot$. For the reasons discussed above, we do not present bulk viscosity timescales for the uniform density star.

Given the form of their eigenfunctions, it seems reasonable to expect that some of the unstable hybrid modes might grow on a timescale which is comparable to that of the pure $l_0 = m$ r-modes. For example, the $m = 2$ axial-led hybrids all have $U_2(r) \neq 0$ (see, for example, Figures 1 and 2).

By equation (65), this leads one to expect a non-zero current quadrupole moment δJ_{22}, and this is the multipole moment that dominates the gravitational radiation in the r-modes. Upon closer inspection, however, one finds that this is not the case. In fact, we find that all of the multipoles δJ_{lm} vanish (or nearly vanish) for $l < l_0$, where l_0 is the largest value of l contributing a dominant term to the expansion (23) of δv^a.

In the uniform density star, these multipoles vanish identically. Consider, for example, the $m = 2$, $l_0 = 4$ axial-hybrid with eigenvalue $\kappa = 0.466901$. (See Table 3 and Figure 6) For this mode, $U_2 \propto (7x^3 - 9x^5)$, where $x = (r/R)$. By equation (65), we then find that

$$\delta J_{22} \propto \int_0^1 x^3(7x^3 - 9x^5)dx \equiv 0, \tag{69}$$

and that δJ_{42} is the only non-zero radiation multipole. In general, the only non-zero multipole for an axial- or polar-hybrid mode in the uniform density star is $\delta J_{l_0\,m}$.

That this should be the case is not obvious from the form of our eigenfunctions. However, Lindblom and Ipser's (1998) analytic solutions provide an explanation. Their equations (7.1) and (7.3) reveal that the perturbed gravitational potential, $\delta\Phi$, is a pure spherical harmonic to lowest order in Ω. In particular,

$$\delta\Phi \propto Y^m_{l_0+1}. \tag{70}$$

This implies that the only non-zero current multipole is $\delta J_{l_0\,m}$.

We find a similar result for the polytropic star. Because of the similarity between the modes in the polytrope and the modes in the uniform density star, we find that although the lower l current multipoles do not vanish identically, they very nearly vanish and the radiation is dominated by higher l multipoles.

The fastest growth times we find in the hybrid modes are of order 10^4 seconds (at 10^9K and $\Omega = \sqrt{\pi G\bar{\rho}}$). Thus, the spin-down of a newly formed neutron star will be dominated by the $l_0 = m = 2$ mode with contributions from the $l_0 = m$ pure axial modes with $2 \leq m \lesssim 10$ and from the fastest growing hybrid modes.

7. Discussion

There is substantial uncertainty in the cooling rate of neutron stars, with rapid cooling expected if stars have a quark interior or core, or a kaon or pion condensate. Madsen (1998) suggests that an observation of a young neutron star with a rotation period below $5 - 10$ms would be evidence for a quark interior; but even without rapid cooling, the uncertainty in the superfluid transition temperature would allow a superfluid to form at about $10^{10}K$, killing the instability. The nonaxisymmetric instability has been expected not to play a role in old neutron stars spun up by accretion, because of the high shear viscosity associated with an expected temperature $\leq 10^7K$; but even this is not certain (Andersson, Kokkotas and Stergioulas 1998).

An extension of our numerical method to find modes of rapidly rotating Newtonian models and slowly rotating relativistic models appears feasible. Work is in progress to understand the way in which the modes join the r- and g- modes of stars that are not isentropic (Andersson et al. 1999).

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A. The character of the modes of rotating isentropic stars

For an equilibrium model that is axisymmetric and invariant under parity, one can resolve any degeneracy in the perturbation spectrum to make each discrete mode an eigenstate of parity with angular dependence $e^{im\varphi}$. The following theorem holds.

Theorem 1 *Let* $(\delta\rho, \delta v^a)$ *with* $\delta v^a \neq 0$ *be a discrete normal mode of a uniformly rotating stellar model obeying a one-parameter equation of state. Then the decomposition of the mode into spherical harmonics* Y_l^m *(i.e., into* (l, m) *representations of the rotation group about its center of mass) has* $l = m$ *as the lowest contributing value of* l*, when* $m \neq 0$*; and has 0 or 1 as the lowest contributing value of* l*, when* $m = 0$*.*

In Sect. III, we designate non-axisymmetric modes with parity $(-1)^m$ "polar-led hybrids", and non-axisymmetric modes with parity $(-1)^{m+1}$ "axial-led hybrids," and briefly discuss the $m = 0$ case.

Note that the theorem holds for p-modes as well as for the rotational modes that are our main concern. A p-mode is determined by its density perturbation and is therefore dominantly polar in character regardless of its parity. For a rotational mode, however, the lowest l term in its velocity perturbation is at least comparable in magnitude to the other contributing terms.

We prove the theorem separately for each parity class.

A.1. Axial-Led Hybrids with $m > 0$

Let l be the smallest value of l' for which $U_{l'} \neq 0$ in the spherical harmonic expansion (23) of the perturbed velocity field δv^a. The axial parity of δv^a, $(-1)^{l+1}$, and the vanishing of Y_l^m for

$l < m$ implies $l \geq m$. That the mode is axial-led means $W_{l'} = 0$ and $V_{l'} = 0$ for $l' \leq l$. We show by contradiction that $l = m$.

Suppose $l \geq m+1$. From equation (38), $\int q^r Y_l^{*m} d\Omega = 0$, we have

$$[\tfrac{1}{2}\kappa l(l+1) - m]U_l = lQ_{l+1}[W_{l+1} + (l+2)V_{l+1}], \tag{A1}$$

and from equation (39) with l replaced by $l-1$, $\int q^\theta Y_{l-1}^{*m} d\Omega = 0$, we have

$$Q_{l+1}\left[(l+2)V'_{l+1} + W'_{l+1}\right] = \left\{[m + \tfrac{1}{2}\kappa(l+1)]U'_l + m(l+1)\frac{U_l}{r}\right\}. \tag{A2}$$

These two equations, together imply that

$$U'_l + \frac{l}{r}U_l = 0,$$

or

$$U_l = Kr^{-l},$$

which is singular at $r = 0$.

A.2. Axial-Led Hybrids with $m = 0$

Let $m = 0$ and let l be the smallest value of l' for which $U_{l'} \neq 0$ in the spherical harmonic expansion (23) of the perturbed velocity field δv^a. Since $\nabla_a Y_0^0 = 0$, the mode vanishes unless $l \geq 1$. That the mode is axial-led means $W_{l'} = 0$ and $V_{l'} = 0$ for $l' \leq l$. We show by contradiction that $l = 1$.

Suppose $l \geq 2$. Then $\int q^\varphi Y_{l-2}^{*0} d\Omega = 0$ becomes,

$$U'_l + \frac{l}{r}U_l = 0, \tag{A3}$$

or

$$U_l = Kr^{-l},$$

which is singular at $r = 0$.

A.3. Polar-Led Hybrids with $m > 0$

Let l be the smallest value of l' for which $W_{l'} \neq 0$ or $V_{l'} \neq 0$ in the spherical harmonic expansion (23) of the perturbed velocity field δv^a. The polar parity of δv^a, $(-1)^l$, and the vanishing of Y_l^m for $l < m$ implies $l \geq m$. That the mode is polar-led means $U_{l'} = 0$ for $l' \leq l$. We show by contradiction that $l = m$.

Suppose $l \geq m+1$. Then $\int q^r Y^{*m}_{l-1} d\Omega = 0$ becomes

$$W_l + (l+1)V_l = 0, \tag{A4}$$

and $\int q^\varphi Y^{*m}_{l-1} d\Omega = 0$ becomes,

$$\begin{aligned} 0 = \;& -\left\{[\tfrac{1}{2}\kappa(l+1)+m]V_l' + m(l+1)\tfrac{V_l}{r} - \tfrac{1}{2}\kappa(l+1)\tfrac{W_l}{r}\right\} \\ & +(l+2)Q_{l+1}\left[U_{l+1}' + (l+1)\tfrac{U_{l+1}}{r}\right]. \end{aligned} \tag{A5}$$

These two equations, together imply that

$$-[\tfrac{1}{2}\kappa(l+1)+m]\left[V_l' + (l+1)\frac{V_l}{r}\right] + (l+2)Q_{l+1}\left[U_{l+1}' + (l+1)\frac{U_{l+1}}{r}\right] = 0,$$

or

$$-[\tfrac{1}{2}\kappa(l+1)+m]V_l + (l+2)Q_{l+1}U_{l+1} = Kr^{-(l+1)},$$

which is singular at $r=0$.

A.4. Polar-Led Hybrids with $m=0$

Let $m=0$ and let l be the smallest value of l' for which $W_{l'} \neq 0$ and $V_{l'} \neq 0$ in the spherical harmonic expansion (23) of the perturbed velocity field δv^a. When $l=0$ the mode is automatically polar-led; thus we need only consider the case $l \geq 1$. That the mode is polar-led means $U_{l'} = 0$ for $l' \leq l$. We show by contradiction that $l=1$.

Suppose $l \geq 2$. Then $\int q^r Y^{*0}_{l-1} d\Omega = 0$ becomes

$$W_l + (l+1)V_l = 0, \tag{A6}$$

and $\int q^\varphi Y^{*0}_{l-1} d\Omega = 0$ becomes,

$$-\tfrac{1}{2}\kappa(l+1)\left[V_l' - \frac{W_l}{r}\right] + (l+2)Q_{l+1}\left[U_{l+1}' + (l+1)\frac{U_{l+1}}{r}\right]. = 0 \tag{A7}$$

These two equations, together imply that

$$-\tfrac{1}{2}\kappa(l+1)\left[V_l' + (l+1)\frac{V_l}{r}\right] + (l+2)Q_{l+1}\left[U_{l+1}' + (l+1)\frac{U_{l+1}}{r}\right] = 0,$$

or

$$-\tfrac{1}{2}\kappa(l+1)V_l + (l+2)Q_{l+1}U_{l+1} = Kr^{-(l+1)},$$

which is singular at $r=0$.

B. The algebraic equations governing the hybrid modes to lowest order in Ω.

In this appendix, we make use of the following definitions:

$$a_l \equiv \tfrac{1}{2}\kappa m + (l+1)Q_l^2 - lQ_{l+1}^2 \tag{B1}$$

$$b_l \equiv m^2 - l(l+1)\left(1 - Q_l^2 - Q_{l+1}^2\right) \tag{B2}$$

$$c_l \equiv \tfrac{1}{2}\kappa l(l+1) - m \tag{B3}$$

For reference, we repeat the definitions (32) and (33):

$$\kappa \equiv \frac{(\sigma + m\Omega)}{\Omega} \tag{B4}$$

$$Q_l \equiv \left[\frac{(l+m)(l-m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}} \tag{B5}$$

B.1. Axial Hybrids

For $l = m,\, m+2,\, m+4, \ldots$ the regular series expansions[10] about the center of the star, $r = 0$, are

$$W_{m+j+1}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} w_{j+1,i} \left(\frac{r}{R}\right)^{i} \tag{B6a}$$

$$V_{m+j+1}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} v_{j+1,i} \left(\frac{r}{R}\right)^{i} \tag{B6b}$$

$$U_{m+j}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=1 \\ i \text{ odd}}}^{\infty} u_{j,i} \left(\frac{r}{R}\right)^{i} \tag{B6c}$$

where $j = 0, 2, 4, \ldots$.

The regular series expansions about $r = R$, which satisfy the boundary condition $\Delta p = 0$ are

$$W_{m+j+1}(r) = \sum_{k=1}^{\infty} \tilde{w}_{j+1,k} \left(1 - \frac{r}{R}\right)^{k} \tag{B7a}$$

$$V_{m+j+1}(r) = \sum_{k=0}^{\infty} \tilde{v}_{j+1,k} \left(1 - \frac{r}{R}\right)^{k} \tag{B7b}$$

$$U_{m+j}(r) = \sum_{k=0}^{\infty} \tilde{u}_{j,k} \left(1 - \frac{r}{R}\right)^{k} \tag{B7c}$$

[10] We present the form of the series expansions for $U_l(r)$ for reference; however, we do not need these series since we eliminate the $U_l(r)$ using equation (38).

where $j = 0, 2, 4, \ldots$.

These series expansions must agree in the interior of the star. We impose the matching condition that the series (B6) truncated at $i_{\max}$ be equal at the point $r = r_0$ to the corresponding series (B7) truncated at $k_{\max}$. That is,

$$0 = \left(\frac{r_0}{R}\right)^{m+j} \sum_{\substack{i=1 \\ i \text{ odd}}}^{i_{\max}} w_{j+1,i} \left(\frac{r_0}{R}\right)^i - \sum_{k=1}^{k_{\max}} \tilde{w}_{j+1,k} \left(1 - \frac{r_0}{R}\right)^k \tag{B8a}$$

$$0 = \left(\frac{r_0}{R}\right)^{m+j} \sum_{\substack{i=1 \\ i \text{ odd}}}^{i_{\max}} v_{j+1,i} \left(\frac{r_0}{R}\right)^i - \sum_{k=0}^{k_{\max}} \tilde{v}_{j+1,k} \left(1 - \frac{r_0}{R}\right)^k \tag{B8b}$$

When we substitute (B6) and (44) into (42), the coefficient of $(r/R)^{m+j+i}$ in the resulting equation is

$$0 = (m+j+i+1)w_{j+1,i} + \sum_{\substack{s=1 \\ s \text{ odd}}}^{i-2} \pi_s \, w_{j+1,i-s-1} - (m+j+1)(m+j+2)v_{j+1,i} \tag{B9}$$

Similarly, when we substitute (B7) and (45) into (42), the coefficient of $[1-(r/R)]^k$ in the resulting equation is

$$0 = (k+1)\left[\tilde{w}_{j+1,k} - \tilde{w}_{j+1,k+1}\right] + \sum_{s=0}^{k} \left(\tilde{\pi}_{s-1} - \tilde{\pi}_{s-2}\right) \tilde{w}_{j+1,k-s+1} - (m+j+2)(m+j+1)\tilde{v}_{j+1,k} \tag{B10}$$

where we have defined $\tilde{\pi}_{-2} \equiv 0 \equiv \tilde{w}_{j+1,0}$.

When we use (38) to eliminate the $U_l(r)$ from (40) and then substitute for the $W_{l\pm1}(r)$ and $V_{l\pm1}(r)$ using (B6), the coefficient of $(r/R)^{m+j+i}$ in the resulting equation is

$$\begin{aligned}
0 = \; & (i+1)(m+j-2)(m+j-1)Q_{m+j}Q_{m+j-1}Q_{m+j-2}c_{m+j}c_{m+j+2} \\
& \qquad \times \left[w_{j-3,i+4} - (m+j-3)v_{j-3,i+4} \right] \\
& - Q_{m+j}c_{m+j+2}\Big\{ (i+1)(m+j-2)^2 Q_{m+j-1}^2 c_{m+j} + \tfrac{1}{2}\kappa(m+j-1)c_{m+j-2}c_{m+j} \\
& \qquad + (m+j+1)\left[(m+j+i)a_{m+j} + b_{m+j}\right] c_{m+j-2} \Big\} w_{j-1,i+2} \\
& + Q_{m+j}c_{m+j+2}\Big\{ \left[\tfrac{1}{2}\kappa(m+j-1)(m+j+i) - (i+1)m\right] c_{m+j-2}c_{m+j} \\
& \qquad + (m+j+1)(m+j-1)\left[(m+j+i)a_{m+j} + b_{m+j}\right] c_{m+j-2} \\
& \qquad - (i+1)(m+j)(m+j-2)^2 Q_{m+j-1}^2 c_{m+j} \Big\} v_{j-1,i+2}
\end{aligned}$$

$$
\begin{aligned}
&+ Q_{m+j+1}c_{m+j-2}\Bigg\{\tfrac{1}{2}\kappa(m+j+2)c_{m+j}c_{m+j+2} + (m+j)\left[(m+j+i)a_{m+j} + b_{m+j}\right]c_{m+j+2} \\
&\qquad - (2m+2j+i+2)(m+j+3)^2 Q_{m+j+2}^2 c_{m+j}\Bigg\} w_{j+1,i} \\
&+ Q_{m+j+1}c_{m+j-2}\Bigg\{(m+j+2)(m+j)\left[(m+j+i)a_{m+j} + b_{m+j}\right]c_{m+j+2} \\
&\qquad - \left[\tfrac{1}{2}\kappa(m+j+2)(m+j+i) + m(2m+2j+i+2)\right]c_{m+j}c_{m+j+2} \\
&\qquad + (2m+2j+i+2)(m+j+3)^2(m+j+1)Q_{m+j+2}^2 c_{m+j}\Bigg\} v_{j+1,i} \\
&+ (2m+2j+i+2)(m+j+3)(m+j+2)Q_{m+j+3}Q_{m+j+2}Q_{m+j+1}c_{m+j-2}c_{m+j} \\
&\qquad\qquad \times \left[w_{j+3,i-2} + (m+j+4)v_{j+3,i-2}\right]
\end{aligned}
\tag{B11}
$$

When we use (38) to eliminate the $U_l(r)$ from (40) and then substitute for the $W_{l\pm1}(r)$ and $V_{l\pm1}(r)$ using (B7), the coefficient of $[1-(r/R)]^k$ in the resulting equation is

$$
\begin{aligned}
0 \quad = \quad & -(m+j-k-1)(m+j-1)(m+j-2)Q_{m+j}Q_{m+j-1}Q_{m+j-2}c_{m+j}c_{m+j+2} \\
&\qquad\qquad \times \left[\tilde{w}_{j-3,k} - (m+j-3)\tilde{v}_{j-3,k}\right] \\
& - (k+1)(m+j-1)(m+j-2)Q_{m+j}Q_{m+j-1}Q_{m+j-2}c_{m+j}c_{m+j+2} \\
&\qquad\qquad \times \left[\tilde{w}_{j-3,k+1} - (m+j-3)\tilde{v}_{j-3,k+1}\right] \\
& + Q_{m+j}c_{m+j+2}\Bigg\{(m+j-k-1)(m+j-2)^2 Q_{m+j-1}^2 c_{m+j} - \tfrac{1}{2}\kappa(m+j-1)c_{m+j-2}c_{m+j} \\
&\qquad - (m+j+1)\left(b_{m+j} + ka_{m+j}\right)c_{m+j-2}\Bigg\}\tilde{w}_{j-1,k} \\
& + (k+1)Q_{m+j}c_{m+j+2}\Bigg\{(m+j-2)^2 Q_{m+j-1}^2 c_{m+j} + (m+j+1)a_{m+j}c_{m+j-2}\Bigg\}\tilde{w}_{j-1,k+1} \\
& + Q_{m+j}c_{m+j+2}\Bigg\{(m+j-k-1)(m+j-2)^2(m+j)Q_{m+j-1}^2 c_{m+j} \\
&\qquad + \left[\tfrac{1}{2}\kappa k(m+j-1) + m(m+j-k-1)\right]c_{m+j-2}c_{m+j} \\
&\qquad + (m+j+1)(m+j-1)\left(b_{m+j} + ka_{m+j}\right)c_{m+j-2}\Bigg\}\tilde{v}_{j-1,k} \\
& + (k+1)Q_{m+j}c_{m+j+2}\Bigg\{(m+j)(m+j-2)^2 Q_{m+j-1}^2 c_{m+j} \\
&\qquad + \left[m - \tfrac{1}{2}\kappa(m+j-1)\right]c_{m+j-2}c_{m+j}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\left. - (m+j+1)(m+j-1)a_{m+j}c_{m+j-2} \right\}\tilde{v}_{j-1,k+1} \\
&+ Q_{m+j+1}c_{m+j-2}\Big\{ (m+j)\left(b_{m+j}+ka_{m+j}\right)c_{m+j+2} + \tfrac{1}{2}\kappa(m+j+2)c_{m+j}c_{m+j+2} \\
&\qquad\left. - (m+j+k+2)(m+j+3)^2 Q^2_{m+j+2}c_{m+j} \right\}\tilde{w}_{j+1,k} \\
&+ (k+1)Q_{m+j+1}c_{m+j-2}\Big\{ -(m+j)a_{m+j}c_{m+j+2} + (m+j+3)^2 Q^2_{m+j+2}c_{m+j} \Big\}\tilde{w}_{j+1,k+1} \\
&+ Q_{m+j+1}c_{m+j-2}\Big\{ (m+j+2)(m+j)\left(b_{m+j}+ka_{m+j}\right)c_{m+j+2} \\
&\qquad - \Big[m(m+j+k+2) + \tfrac{1}{2}\kappa k(m+j+2) \Big] c_{m+j}c_{m+j+2} \\
&\qquad\left. + (m+j+k+2)(m+j+3)^2(m+j+1)Q^2_{m+j+2}c_{m+j} \right\}\tilde{v}_{j+1,k} \\
&+ (k+1)Q_{m+j+1}c_{m+j-2}\Big\{ -(m+j+2)(m+j)a_{m+j}c_{m+j+2} \\
&\qquad + \Big[\tfrac{1}{2}\kappa(m+j+2) + m \Big] c_{m+j}c_{m+j+2} \\
&\qquad\left. - (m+j+3)^2(m+j+1)Q^2_{m+j+2}c_{m+j} \right\}\tilde{v}_{j+1,k+1} \\
&+ (m+j+k+2)(m+j+3)(m+j+2)Q_{m+j+3}Q_{m+j+2}Q_{m+j+1}c_{m+j-2}c_{m+j} \\
&\qquad\qquad\qquad \times \Big[\tilde{w}_{j+3,k} + (m+j+4)\tilde{v}_{j+3,k} \Big] \\
&- (k+1)(m+j+3)(m+j+2)Q_{m+j+3}Q_{m+j+2}Q_{m+j+1}c_{m+j-2}c_{m+j} \\
&\qquad\qquad\qquad \times \Big[\tilde{w}_{j+3,k+1} + (m+j+4)\tilde{v}_{j+3,k+1} \Big]
\end{aligned} \tag{B12}
$$

The equations (B8) through (B12) make up the algebraic system (46) for eigenvalues of the axial-led hybrid modes. One truncates the angular and radial series expansions at indices $j_{\max}$, $i_{\max}$ and $k_{\max}$ and constructs the matrix A by keeping the appropriate number of equations for the number of unknown coefficients $w_{j+1,i}$, $v_{j+1,i}$, $\tilde{w}_{j+1,k}$ and $\tilde{v}_{j+1,k}$. In following this procedure, however, one must be aware of the following subtlety in the equations.

For each $q \equiv j+i$ the set of equations

$$
\begin{array}{lll}
\text{(B9)} & \text{with } \ i=1 \ \text{ and } \ j=q-1, \ \text{ and} \\
\text{(B11)} & \text{for all } \ i=1,3,\ldots,q \ \text{ with } \ j=q-i
\end{array}
$$

can be shown to be linearly dependent for arbitrary κ and for any equilibrium stellar model. For example, taking the simplest case of $q=1$, one can show that equation (B9) with $i=1$ and $j=0$

becomes

$$0 = (m+2)\left[w_{1,1} - (m+1)v_{1,1}\right]$$

while equation (B11) with $i = 1$ and $j = 0$ becomes

$$\begin{aligned} 0 = \; & Q_{m+1}c_{m-2}\left\{\tfrac{1}{2}\kappa(m+2)c_m c_{m+2} + m\left[(m+1)a_m + b_m\right] - (2m+3)(m+3)^2 Q_{m+2}^2 c_m\right\} \\ & \times \left[w_{1,1} - (m+1)v_{1,1}\right]. \end{aligned}$$

This problem can be solved by eliminating one of these equations from the subset for each q (for example, equation (B11) with $i = 1$). Thus, to properly construct the algebraic system (46) we use, for all $j = 0, 2, \ldots, j_{\max}$, the equations

(B8a)		
(B8b)		
(B9)	with	$i = 1, 3, \ldots, i_{\max}$
(B10)	with	$k = 0, 1, \ldots, k_{\max} - 1$
(B11)	with	$i = 3, 5, \ldots, i_{\max}$
(B12)	with	$k = 0, 1, \ldots, k_{\max} - 1$.

B.2. Polar Hybrids

For $l = m,\ m+2,\ m+4, \ldots$ the regular series expansions[11] about the center of the star, $r = 0$, are

$$W_{m+j}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=0 \\ i \text{ even}}}^{\infty} w_{j,i} \left(\frac{r}{R}\right)^{i} \tag{B13a}$$

$$V_{m+j}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=0 \\ i \text{ even}}}^{\infty} v_{j,i} \left(\frac{r}{R}\right)^{i} \tag{B13b}$$

$$U_{m+j+1}(r) = \left(\frac{r}{R}\right)^{m+j} \sum_{\substack{i=2 \\ i \text{ even}}}^{\infty} u_{j+1,i} \left(\frac{r}{R}\right)^{i} \tag{B13c}$$

where $j = 0, 2, 4, \ldots$.

The regular series expansions about $r = R$, which satisfy the boundary condition $\Delta p = 0$ are

$$W_{m+j}(r) = \sum_{k=1}^{\infty} \tilde{w}_{j,k} \left(1 - \frac{r}{R}\right)^{k} \tag{B14a}$$

[11] We present the form of the series expansions for $U_l(r)$ for reference; however, we do not need these series since we eliminate the $U_l(r)$ using equation (38).

$$V_{m+j}(r) = \sum_{k=0}^{\infty} \tilde{v}_{j,k} \left(1 - \frac{r}{R}\right)^k \tag{B14b}$$

$$U_{m+j+1}(r) = \sum_{k=0}^{\infty} \tilde{u}_{j+1,k} \left(1 - \frac{r}{R}\right)^k \tag{B14c}$$

where $j = 0, 2, 4, \ldots$.

These series expansions must agree in the interior of the star. We impose the matching condition that the series (B13) truncated at $i_{\max}$ be equal at the point $r = r_0$ to the corresponding series (B14) truncated at $k_{\max}$. That is,

$$0 = \left(\frac{r_0}{R}\right)^{m+j} \sum_{\substack{i=0 \\ i \text{ even}}}^{i_{\max}} w_{j,i} \left(\frac{r_0}{R}\right)^i - \sum_{k=1}^{k_{\max}} \tilde{w}_{j,k} \left(1 - \frac{r_0}{R}\right)^k \tag{B15a}$$

$$0 = \left(\frac{r_0}{R}\right)^{m+j} \sum_{\substack{i=0 \\ i \text{ even}}}^{i_{\max}} v_{j,i} \left(\frac{r_0}{R}\right)^i - \sum_{k=0}^{k_{\max}} \tilde{v}_{j,k} \left(1 - \frac{r_0}{R}\right)^k \tag{B15b}$$

When we substitute (B13) and (44) into (42), the coefficient of $(r/R)^{m+j+i}$ in the resulting equation is

$$0 = (m+j+i+1)w_{j,i} + \sum_{\substack{s=0 \\ s \text{ even}}}^{i-2} \pi_{s+1}\, w_{j,i-s-2} - (m+j)(m+j+1)v_{j,i} \tag{B16}$$

Similarly, when we substitute (B14) and (45) into (42), the coefficient of $[1 - (r/R)]^k$ in the resulting equation is

$$0 = (k+1)\left[\tilde{w}_{j,k} - \tilde{w}_{j,k+1}\right] + \sum_{s=0}^{k} \left(\tilde{\pi}_{s-1} - \tilde{\pi}_{s-2}\right) \tilde{w}_{j,k-s+1} - (m+j)(m+j+1)\tilde{v}_{j,k} \tag{B17}$$

where we have defined $\tilde{\pi}_{-2} \equiv 0 \equiv \tilde{w}_{j,0}$.

When we use (38) to eliminate the $U_l(r)$ from (39) and then substitute for the $W_{l\pm1}(r)$ and $V_{l\pm1}(r)$ using (B13), the coefficient of $(r/R)^{m+j+i}$ in the resulting equation is

$$\begin{aligned} 0 \;=\; & -im(m+j-1)Q_{m+j}Q_{m+j-1}c_{m+j+1}\Big[w_{j-2,i+2} - (m+j-2)v_{j-2,i+2}\Big] \\ & + \Big\{(m+j-1)Q^2_{m+j}[(i+1)m - \tfrac{1}{2}\kappa(m+j-1)(m+j+i)]c_{m+j+1} \\ & \qquad + \Big[(m+j+i)\left(1 - Q^2_{m+j} - Q^2_{m+j+1}\right) + \tfrac{1}{2}\kappa m\Big]\, c_{m+j-1}c_{m+j+1} \\ & \qquad - (m+j+2)Q^2_{m+j+1}[m(2m+2j+i+2) + \tfrac{1}{2}\kappa(m+j+2)(m+j+i)]c_{m+j-1}\Big\} w_{j,i} \end{aligned}$$

$$
\begin{aligned}
&+\Big\{(m+j-1)(m+j+1)Q^2_{m+j}[(i+1)m-\tfrac{1}{2}\kappa(m+j-1)(m+j+i)]c_{m+j+1} \\
&\qquad -\Big[m^2+(m+j+i)a_{m+j}\Big]c_{m+j-1}c_{m+j+1} \\
&\qquad +(m+j)(m+j+2)Q^2_{m+j+1} \\
&\qquad\qquad\qquad \times\Big[m(2m+2j+i+2)+\tfrac{1}{2}\kappa(m+j+2)(m+j+i)\Big]c_{m+j-1}\Big\}v_{j,i} \\
&+Q_{m+j+2}Q_{m+j+1}\left[m(m+j+i)+m(m+j+1)(2m+2j+i+2)\right]c_{m+j-1} \\
&\qquad\qquad\qquad \times\Big[w_{j+2,i-2}+(m+j+3)v_{j+2,i-2}\Big]
\end{aligned} \tag{B18}
$$

When we use (38) to eliminate the $U_l(r)$ from (39) and then substitute for the $W_{l\pm1}(r)$ and $V_{l\pm1}(r)$ using (B14), the coefficient of $[1-(r/R)]^k$ in the resulting equation is

$$
\begin{aligned}
0 \;=\;& m(m+j-1)(m+j-k)Q_{m+j}Q_{m+j-1}c_{m+j+1}\Big[\tilde{w}_{j-2,k}-(m+j-2)\tilde{v}_{j-2,k}\Big] \\
&+(k+1)m(m+j-1)Q_{m+j}Q_{m+j-1}c_{m+j+1}\Big[\tilde{w}_{j-2,k+1}-(m+j-2)\tilde{v}_{j-2,k+1}\Big] \\
&+\Big\{-(m+j-1)Q^2_{m+j}[(\tfrac{1}{2}\kappa k+m)(m+j-1)-km]c_{m+j+1} \\
&\qquad +\Big[\tfrac{1}{2}\kappa m+k\Big(1-Q^2_{m+j}-Q^2_{m+j+1}\Big)\Big]c_{m+j-1}c_{m+j+1} \\
&\qquad -(m+j+2)Q^2_{m+j+1}[(\tfrac{1}{2}\kappa k+m)(m+j+2)+km]c_{m+j-1}\Big\}\tilde{w}_{j,k} \\
&-(k+1)\Big\{(m+j-1)Q^2_{m+j}[m-\tfrac{1}{2}\kappa(m+j-1)]c_{m+j+1} \\
&\qquad\qquad +\Big(1-Q^2_{m+j}-Q^2_{m+j+1}\Big)c_{m+j-1}c_{m+j+1} \\
&\qquad\qquad -(m+j+2)Q^2_{m+j+1}[m+\tfrac{1}{2}\kappa(m+j+2)]c_{m+j-1}\Big\}\tilde{w}_{j,k+1} \\
&+\Big\{-(m+j-1)(m+j+1)Q^2_{m+j}[(\tfrac{1}{2}\kappa k+m)(m+j-1)-km]c_{m+j+1} \\
&\qquad -\Big(m^2+ka_{m+j}\Big)c_{m+j-1}c_{m+j+1} \\
&\qquad +(m+j)(m+j+2)Q^2_{m+j+1}[(\tfrac{1}{2}\kappa k+m)(m+j+2)+km]c_{m+j-1}\Big\}\tilde{v}_{j,k} \\
&+(k+1)\Big\{-(m+j-1)(m+j+1)Q^2_{m+j}[m-\tfrac{1}{2}\kappa(m+j-1)]c_{m+j+1} \\
&\qquad\qquad +a_{m+j}c_{m+j-1}c_{m+j+1}
\end{aligned}
$$

$$
\begin{aligned}
&- (m+j)(m+j+2)Q^2_{m+j+1}[m+\tfrac{1}{2}\kappa(m+j+2)]c_{m+j-1}\Big\}\tilde{v}_{j,k+1} \\
&+ m(m+j+2)(m+j+k+1)Q_{m+j+2}Q_{m+j+1}c_{m+j-1}\Big[\tilde{w}_{j+2,k}+(m+j+3)\tilde{v}_{j+2,k}\Big] \\
&- (k+1)m(m+j+2)Q_{m+j+2}Q_{m+j+1}c_{m+j-1}\Big[\tilde{w}_{j+2,k+1}+(m+j+3)\tilde{v}_{j+2,k+1}\Big] \qquad \text{(B19)}
\end{aligned}
$$

The equations (B15) through (B19) make up the algebraic system (46) for eigenvalues of the polar-led hybrid modes. As in the case of the axial-led hybrids, one truncates the angular and radial series expansions at indices $j_{\max}$, $i_{\max}$ and $k_{\max}$ and constructs the matrix A by keeping the appropriate number of equations for the number of unknown coefficients $w_{j,i}$, $v_{j,i}$, $\tilde{w}_{j,k}$ and $\tilde{v}_{j,k}$.

We, again, find that certain subsets of these equations are linearly dependent for arbitrary κ and for any equilibrium stellar model. For all j, it can be shown that both equation (B16) with $i=0$ and equation (B18) with $i=0$ are proportional to

$$0 = [w_{j,0} - (m+j)v_{j,0}]\,.$$

This problem can, again, be solved by eliminating, for example, equation (B18) with $i=0$ for all j. Thus, to properly construct the algebraic system (46) we use, for all $j=0,2,\ldots,j_{\max}$, the equations

(B15a)		
(B15b)		
(B16)	with	$i=0,2,\ldots,i_{\max}$
(B17)	with	$k=0,1,\ldots,k_{\max}-1$
(B18)	with	$i=2,4,\ldots,i_{\max}$
(B19)	with	$k=0,1,\ldots,k_{\max}-1$.

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Fig. 1.— All of the non-zero coefficients $W_l(r)$, $V_l(r)$, $U_l(r)$ of the spherical harmonic expansion (23) for a particular $m = 2$ axial-led hybrid mode of the uniform density star. The mode has eigenvalue $\kappa_0 = -0.763337$. Note that the largest value of l that appears in the expansion (23) is $l_0 = 4$ and that the functions $W_l(r)$, $V_l(r)$ and $U_l(r)$ are low order polynomials in (r/R). (See Table 3.) The mode is normalized so that $V_2(r = R) = 1$.

Fig. 2.— The coefficients $W_l(r)$, $V_l(r)$, $U_l(r)$ with $l \leq 6$ of the spherical harmonic expansion (23) for a particular $m = 2$ axial-led hybrid mode of the polytropic star. This is the polytrope mode that corresponds to the uniform density mode displayed in Figure 1. For the polytrope the mode has eigenvalue $\kappa_0 = -1.025883$. The expansion (23) converges rapidly with increasing l and is dominated by the terms with $2 \leq l \leq 4$, i.e., by the terms corresponding to those which are non-zero for the uniform density mode. Observe that the coefficients shown with $4 < l \leq 6$ are an order of magnitude smaller than those with $2 \leq l \leq 4$. Those with $l > 6$ are smaller still and are not displayed here. The mode is, again, normalized so that $V_2(r = R) = 1$.

Fig. 3.— The functions $W_l(r)$ with $l \leq 6$ for a particular $m = 1$ polar-led hybrid mode. For the uniform density star this mode has eigenvalue $\kappa_0 = 1.509941$ and $W_1 = -x + x^3$ $(x = r/R)$ is the only non-vanishing $W_l(r)$ (see Table 2). The corresponding mode of the polytropic star has eigenvalue $\kappa_0 = 1.412999$. Observe that $W_1(r)$ for the polytrope, which has been constructed from its power series expansions about $r = 0$ and $r = R$, is similar, though not identical, to the corresponding $W_1(r)$ for the uniform density star. Observe also that the functions $W_l(r)$ with $l > 1$ for the polytrope are more than an order of magnitude smaller than $W_1(r)$ and become smaller with increasing l. ($W_5(r)$ is virtually indistinguishable from the (r/R) axis.)

Fig. 4.— The functions $V_l(r)$ with $l \leq 6$ for the same mode as in Figure 3.

Fig. 5.— The functions $U_l(r)$ with $l \leq 6$ for the same mode as in Figure 3.

Fig. 6.— The functions $U_l(r)$ with $l \leq 7$ for a particular $m = 2$ axial-led hybrid mode. For the uniform density star this mode has eigenvalue $\kappa_0 = 0.466901$ and $U_2(r)$ and $U_4(r)$ are the only non-vanishing $U_l(r)$. (See Table 3 for their explicit forms.) The corresponding mode of the polytropic star has eigenvalue $\kappa_0 = 0.517337$. Observe that $U_2(r)$ and $U_4(r)$ for the polytrope, which have been constructed from their power series expansions about $r = 0$ and $r = R$, are similar, though not identical, to the corresponding functions for the uniform density star. Observe also that the $U_6(r)$ is more than an order of magnitude smaller than $U_2(r)$ and $U_4(r)$.

Fig. 7.— The functions $W_l(r)$ with $l \leq 7$ for the same mode as in Figure 6.

Fig. 8.— The functions $V_l(r)$ with $l \leq 7$ for the same mode as in Figure 6.

Fig. 9.— The functions $U_l(r)$ with $l \leq 8$ for a particular $m = 2$ axial-led hybrid mode. For the uniform density star this mode has eigenvalue $\kappa_0 = 0.359536$ and $U_2(r)$, $U_4(r)$ and $U_6(r)$ are the only non-vanishing $U_l(r)$. (See Table 3 for their explicit forms.) The corresponding mode

of the polytropic star has eigenvalue $\kappa_0 = 0.421678$. Observe that $U_2(r)$, $U_4(r)$ and $U_6(r)$ for the polytrope, which have been constructed from their power series expansions about $r = 0$ and $r = R$, are similar, though not identical, to the corresponding functions for the uniform density star. Observe also that $U_8(r)$ is more than an order of magnitude smaller than $U_2(r)$, $U_4(r)$ and $U_6(r)$.

Fig. 10.— The functions $W_l(r)$ with $l \leq 8$ for the same mode as in Figure 9.

Fig. 11.— The functions $V_l(r)$ with $l \leq 8$ for the same mode as in Figure 9.

TABLE 1

Axial-Hybrid Eigenfunctions[a] with $m = 1$ for Uniform Density Stars.

l_0[b]	κ_0	$U_1(r)$	$U_3(r)$	$U_5(r)$	$W_2(r)$	$W_4(r)$	$V_2(r)$	$V_4(r)$
1	1.000000	x^2	0	0	0	0	0	0
3	-0.820009	$0.368581(5x^2 - 7x^4)$	$-0.646064x^4$	0	$-3(x - x^3)$	0	$-1.5x + 2.5x^3$	0
	0.611985*	$1.728851(5x^2 - 7x^4)$	$1.431460x^4$	0	$-3(x - x^3)$	0	$-1.5x + 2.5x^3$	0
	1.708024	$-0.947454(5x^2 - 7x^4)$	$0.413567x^4$	0	$-3(x - x^3)$	0	$-1.5x + 2.5x^3$	0
5	-1.404217	$-0.279018(8.75x^2 - 31.5x^4 + 24.75x^6)$	$0.583566(9x^4 - 11x^6)$	$-0.525092x^6$	$5.25x^2 - 13.5x^4 + 8.25x^6$	$-0.490203(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$-0.490203(5x^4 - 7x^6)$
	-0.537334	$-0.436353(8.75x^2 - 31.5x^4 + 24.75x^6)$	$0.188398(9x^4 - 11x^6)$	$0.397943x^6$	$5.25x^2 - 13.5x^4 + 8.25x^6$	$0.152553(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$0.152553(5x^4 - 7x^6)$
	0.440454*	$-1.198867(8.75x^2 - 31.5x^4 + 24.75x^6)$	$-0.462550(9x^4 - 11x^6)$	$-0.663736x^6$	$5.25x^2 - 13.5x^4 + 8.25x^6$	$0.157465(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$0.157465(5x^4 - 7x^6)$
	1.306079	$2.191660(8.75x^2 - 31.5x^4 + 24.75x^6)$	$0.387296(9x^4 - 11x^6)$	$-0.792009x^6$	$5.25x^2 - 13.5x^4 + 8.25x^6$	$0.623029(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$0.623029(5x^4 - 7x^6)$
	1.861684	$0.778500(8.75x^2 - 31.5x^4 + 24.75x^6)$	$-0.326313(9x^4 - 11x^6)$	$0.168645x^6$	$5.25x^2 - 13.5x^4 + 8.25x^6$	$-0.192134(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$-0.192134(5x^4 - 7x^6)$

[a] The eigenfunctions are normalized so that $V_2 = 1$ at the surface of the star, $x = 1$. Here $x = (r/R)$.

[b] $l_0 - m + 1 = l_0$ is the maximum value of l in the spherical harmonic expansion of δv^a. Observe that this $l = l_0$ term is always axial.

TABLE 2

POLAR-HYBRID EIGENFUNCTIONS[a] WITH $m = 1$ FOR UNIFORM DENSITY STARS.

l_0[b]	κ_0	$W_1(r)$	$W_3(r)$	$V_1(r)$	$V_3(r)$	$U_2(r)$	$U_4(r)$
2	-0.176607	$-x + x^3$	0	$-x + 2x^3$	0	$-0.876991x^3$	0
	1.509941	$-x + x^3$	0	$-x + 2x^3$	0	$0.380087x^3$	0
4	-1.183406	$1.25x - 3.5x^3 + 2.25x^5$	$-1.585327(6x^3 - 6x^5)$	$1.25x - 7x^3 + 6.75x^5$	$-1.585327(2x^3 - 3x^5)$	$0.813707(7x^3 - 9x^5)$	$-0.904110x^5$
	-0.068189	$1.25x - 3.5x^3 + 2.25x^5$	$0.100030(6x^3 - 6x^5)$	$1.25x - 7x^3 + 6.75x^5$	$0.100030(2x^3 - 3x^5)$	$0.398091(7x^3 - 9x^5)$	$0.435309x^5$
	1.045597	$1.25x - 3.5x^3 + 2.25x^5$	$0.331793(6x^3 - 6x^5)$	$1.25x - 7x^3 + 6.75x^5$	$0.331793(2x^3 - 3x^5)$	$-0.016993(7x^3 - 9x^5)$	$-0.256819x^5$
	1.805998	$1.25x - 3.5x^3 + 2.25x^5$	$-0.343160(6x^3 - 6x^5)$	$1.25x - 7x^3 + 6.75x^5$	$-0.343160(2x^3 - 3x^5)$	$-0.300378(7x^3 - 9x^5)$	$0.147226x^5$

[a] The eigenfunctions are normalized so that $V_1 = 1$ at the surface of the star, $x = 1$. Here $x = (r/R)$.

[b] $l_0 - m + 1 = l_0$ is the maximum value of l in the spherical harmonic expansion of δv^a. Observe that this $l = l_0$ term is always axial.

TABLE 3

AXIAL-HYBRID EIGENFUNCTIONS[a] WITH $m = 2$ FOR UNIFORM DENSITY STARS.

$(l_0 - 1)$[b]	κ_0	$U_2(r)$	$U_4(r)$	$U_6(r)$	$W_3(r)$	$W_5(r)$	$V_3(r)$	$V_5(r)$
1	0.666667	x^3	0	0	0	0	0	0
3	-0.763337	$0.352414(7x^3 - 9x^5)$	$-0.679569x^5$	0	$-6(x^3 - x^5)$	0	$-2x^3 + 3x^5$	0
	0.466901*	$2.522714(7x^3 - 9x^5)$	$2.452800x^5$	0	$-6(x^3 - x^5)$	0	$-2x^3 + 3x^5$	0
	1.496436	$-0.607406(7x^3 - 9x^5)$	$0.504964x^5$	0	$-6(x^3 - x^5)$	0	$-2x^3 + 3x^5$	0
5	-1.308000	$-0.510418(7.875x^3 - 24.75x^5 + 17.875x^7)$	$0.634277(11x^5 - 13x^7)$	$-0.639609x^7$	$13.5x^3 - 33x^5 + 19.5x^7$	$-1.138387(15x^5 - 15x^7)$	$4.5x^3 - 16.5x^5 + 13x^7$	$-1.138387(3x^5 - 4x^7)$
	-0.509994	$-0.856581(7.875x^3 - 24.75x^5 + 17.875x^7)$	$0.188642(11x^5 - 13x^7)$	$0.455827x^7$	$13.5x^3 - 33x^5 + 19.5x^7$	$0.349918(15x^5 - 15x^7)$	$4.5x^3 - 16.5x^5 + 13x^7$	$0.349918(3x^5 - 4x^7)$
	0.359536*	$-3.281679(7.875x^3 - 24.75x^5 + 17.875x^7)$	$-0.769879(11x^5 - 13x^7)$	$-1.103800x^7$	$13.5x^3 - 33x^5 + 19.5x^7$	$0.370022(15x^5 - 15x^7)$	$4.5x^3 - 16.5x^5 + 13x^7$	$0.370022(3x^5 - 4x^7)$
	1.153058*	$2.072212(7.875x^3 - 24.75x^5 + 17.875x^7)$	$0.102912(11x^5 - 13x^7)$	$-0.573689x^7$	$13.5x^3 - 33x^5 + 19.5x^7$	$0.769719(15x^5 - 15x^7)$	$4.5x^3 - 16.5x^5 + 13x^7$	$0.769719(3x^5 - 4x^7)$
	1.733971*	$0.944346(7.875x^3 - 24.75x^5 + 17.875x^7)$	$-0.423766(11x^5 - 13x^7)$	$0.280929x^7$	$13.5x^3 - 33x^5 + 19.5x^7$	$-0.583914(15x^5 - 15x^7)$	$4.5x^3 - 16.5x^5 + 13x^7$	$-0.583914(3x^5 - 4x^7)$

[a] The eigenfunctions are normalized so that $V_3 = 1$ at the surface of the star, $x = 1$. Here $x = (r/R)$.

[b] $l_0 - m + 1 = l_0 - 1$, where l_0 is the maximum value of l in the spherical harmonic expansion of δv^a. Observe that this $l = l_0$ term is always axial.

TABLE 4

Polar-Hybrid Eigenfunctions[a] with $m = 2$ for Uniform Density Stars.

$(l_0 - 1)^{\mathrm{b}}$	κ_0	$W_2(r)$	$W_4(r)$	$V_2(r)$	$V_4(r)$	$U_3(r)$	$U_5(r)$
2	-0.231925	$-3x^2 + 3x^4$	0	$-1.5x^2 + 2.5x^4$	0	$-0.891544x^4$	0
	1.231925*	$-3x^2 + 3x^4$	0	$-1.5x^2 + 2.5x^4$	0	$0.560825x^4$	0
4	-1.092568	$5.25x^2 - 13.5x^4 + 8.25x^6$	$-0.909581(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$-0.909581(5x^4 - 7x^6)$	$0.872718(9x^4 - 11x^6)$	$-1.093523x^6$
	-0.101790	$5.25x^2 - 13.5x^4 + 8.25x^6$	$0.078913(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$0.078913(5x^4 - 7x^6)$	$0.381215(9x^4 - 11x^6)$	$0.494643x^6$
	0.884249*	$5.25x^2 - 13.5x^4 + 8.25x^6$	$0.176440(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$0.176440(5x^4 - 7x^6)$	$-0.107938(9x^4 - 11x^6)$	$-0.346296x^6$
	1.643443*	$5.25x^2 - 13.5x^4 + 8.25x^6$	$-0.350886(20x^4 - 20x^6)$	$2.625x^2 - 11.25x^4 + 9.625x^6$	$-0.350886(5x^4 - 7x^6)$	$-0.484558(9x^4 - 11x^6)$	$0.342451x^6$

[a] The eigenfunctions are normalized so that $V_2 = 1$ at the surface of the star, $x = 1$. Here $x = (r/R)$.

[b] $l_0 - m + 1 = l_0 - 1$, where l_0 is the maximum value of l in the spherical harmonic expansion of δv^a. Observe that this $l = l_0$ term is always axial.

Table 5. Eigenvalues κ_0[a] for Uniform Density Stars.

$(l_0 - m + 1)$[b]	parity[c]	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
1[d]	a	0.000000	1.000000	0.666667*	0.500000*	0.400000*
2	p	-0.894427	-0.176607	-0.231925	-0.253197	-0.261255
	p	0.894427	1.509941	1.231925*	1.053197*	0.927922*
3	a	-1.309307	-0.820009	-0.763337	-0.718066	-0.680693
	a	0.000000	0.611985*	0.466901*	0.377861*	0.317496*
	a	1.309307	1.708024	1.496436*	1.340205*	1.220340*
4	p	-1.530111	-1.183406	-1.092568	-1.022179	-0.965177
	p	-0.570463	-0.068189	-0.101790	-0.120347	-0.131215
	p	0.570463	1.045597	0.884249*	0.773460*	0.691976*
	p	1.530111	1.805998	1.643443*	1.511923*	1.404416*
5	a	-1.660448	-1.404217	-1.308000	-1.230884	-1.167037
	a	-0.937698	-0.537334	-0.509994	-0.486868	-0.466934
	a	0.000000	0.440454*	0.359536*	0.304044*	0.263530*
	a	0.937698	1.306079	1.153058*	1.040073*	0.952507*
	a	1.660448	1.861684	1.733971*	1.623634*	1.529045*

[a] $\kappa_0\Omega = (\sigma + m\Omega)$ is the mode frequency in the rotating frame to lowest order in Ω. The modes whose frequencies are marked with a $*$ satisfy the condition $\sigma(\sigma + m\Omega) < 0$ and are subject to a gravitational radiation driven instability in the absence of viscous dissipation.

[b] For $m = 0$, this is simply l_0. For the uniform density star, l_0 is the maximum value of l appearing in the spherical harmonic expansion of δv^a.

[c] This denotes the parity class of the mode; a meaning axial-led hybrids, and p meaning polar-led hybrids.

[d] These are the eigenvalues of the pure $l_0 = m$ r-modes. For isentropic stars they are independent of the equation of state and have the value $\kappa_0 = 2/(m+1)$ (or $\kappa_0 = 0$ for $m = 0$) to lowest order in Ω (Papalouizou and Pringle 1978).

Table 6. Eigenvalues κ_0[a] for the $p = K\rho^2$ Polytrope.

$(l_0 - m + 1)$[b]	parity[c]	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
1[d]	a	0.000000	1.000000	0.666667*	0.500000*	0.400000*
2	p	-1.028189	-0.401371	-0.556592	-0.631637	-0.672385
	p	1.028189	1.412999	1.100026*	0.904910*	0.771078*
3	a	-1.358128	-1.032380	-1.025883	-1.014866	-1.002175
	a	0.000000	0.690586*	0.517337*	0.412646*	0.342817*
	a	1.358128	1.613725	1.357781*	1.176745*	1.041683*
4	p	-1.542065	-1.312267	-1.272885	-1.238631	-1.208390
	p	-0.701821	-0.178792	-0.275335	-0.333267	-0.370450
	p	0.701821	1.051525	0.862948*	0.734297*	0.640592*
	p	1.542065	1.726257	1.519573*	1.360560*	1.234698*
5	a	-1.656481	-1.483402	-1.433916	-1.391305	-1.354057
	a	-1.013703	-0.705182	-0.703898	-0.699942	-0.694498
	a	0.000000	0.528102*	0.421678*	0.350192*	0.299055*
	a	1.013703	1.281962	1.104402*	0.974192*	0.874124*
	a	1.656481	1.795734	1.627215*	1.489441*	1.375406*

[a] $\kappa_0\Omega = (\sigma + m\Omega)$ is the mode frequency in the rotating frame to lowest order in Ω. The modes whose frequencies are marked with a $*$ satisfy the condition $\sigma(\sigma + m\Omega) < 0$ and are subject to a gravitational radiation driven instability in the absence of viscous dissipation.

[b] For $m = 0$, this is simply l_0. For the $n = 1$ polytrope, l_0 is the largest value of l that contributes a dominant term to the spherical harmonic expansion of δv^a.

[c] This denotes the parity class of the mode; a meaning axial-led hybrids, and p meaning polar-led hybrids.

[d] These are the eigenvalues of the pure $l_0 = m$ r-modes. For isentropic stars they are independent of the equation of state and have the value $\kappa_0 = 2/(m + 1)$ (or $\kappa_0 = 0$ for $m = 0$) to lowest order in Ω (Papalouizou and Pringle 1978).

Table 7. Dissipative timescales (in seconds) for $m = 1$ axial-hybrid modes[a] at $T = 10^9 K$ and $\Omega = \sqrt{\pi G \bar{\rho}}$.

l_0	n[b]	κ	$\tilde{\tau}_B$[c]	$\tilde{\tau}_S$	$\tilde{\tau}_3$	$\tilde{\tau}_5$
3	0	0.611985	$\cdots$	7.67×10^7	-9.79×10^6	$\cdots$
	1	0.690586	5.86×10^9	9.29×10^7	-1.25×10^8	-1.22×10^{20}
5	0	0.440454	$\cdots$	2.04×10^7	$-\infty$	-2.07×10^{13}
	1	0.528102	2.57×10^9	3.87×10^7	-2.17×10^{10}	-5.75×10^{14}

[a]We present dissipative timescales only for those modes that are unstable to gravitational radiation reaction. None of the $m = 1$ polar-hybrid modes are unstable for low values of l_0.

[b]The polytropic index, n, where $p = K\rho^{1+1/n}$. The n=0 case represents the uniform density equilibrium star.

[c]Dissipation due to bulk viscosity is not meaningful for uniform density stars.

Table 8. Dissipative timescales (in seconds) for $m = 2$ axial-hybrid modes at $T = 10^9 K$ and $\Omega = \sqrt{\pi G \bar{\rho}}$.

$(l_0 - 1)$	n[a]	κ	$\tilde{\tau}_B$[b]	$\tilde{\tau}_S$	$\tilde{\tau}_2$	$\tilde{\tau}_4$	$\tilde{\tau}_6$
1[c]	0	0.666667	$\cdots$	4.46×10^8	-1.56×10^0	$\cdots$	$\cdots$
	1	0.666667	2.0×10^{11}	2.52×10^8	-3.26×10^0	$\cdots$	$\cdots$
3	0	0.466901	$\cdots$	4.10×10^7	$-\infty$	-3.88×10^5	$\cdots$
	1	0.517337	6.43×10^9	6.21×10^7	$< -10^{18}$	-1.85×10^6	-4.97×10^{15}
	0	1.496436	$\cdots$	3.92×10^7	$-\infty$	-5.85×10^9	$\cdots$
	1	1.357781	4.10×10^9	7.18×10^7	$< -10^{19}$	-1.60×10^9	-4.35×10^{19}
5	0	0.359536	$\cdots$	1.34×10^7	$-\infty$	$-\infty$	-1.28×10^{11}
	1	0.421678	2.65×10^9	3.01×10^7	$< -10^{16}$	-2.01×10^9	-1.15×10^{12}
	0	1.153058	$\cdots$	1.32×10^7	$-\infty$	$-\infty$	-3.11×10^{14}
	1	1.104402	2.45×10^9	3.65×10^7	$< -10^{12}$	-1.37×10^{11}	-4.89×10^{14}
	0	1.733971	$\cdots$	1.31×10^7	$-\infty$	$-\infty$	-1.92×10^{21}
	1	1.627215	5.32×10^9	3.44×10^7	$< -10^{19}$	-2.30×10^{15}	-8.33×10^{19}

[a]The polytropic index, n, where $p = K\rho^{1+1/n}$. The n=0 case represents the uniform density equilibrium star.

[b]Dissipation due to bulk viscosity is not meaningful for uniform density stars.

[c]This is the $l_0 = m = 2$ r-mode already studied by Lindblom et al. (1998), Owen et al. (1998), Andersson et al. (1998), Kokkotas and Stergioulas (1998) and Lindblom et al. (1999). The value of the bulk viscosity timescale for this mode is taken from Lindblom et al. (1999) who calculate it self-consistently using an order Ω^2 calculation.

Table 9. Dissipative timescales (in seconds) for $m = 2$ polar-hybrid modes at $T = 10^9 K$ and $\Omega = \sqrt{\pi G\bar{\rho}}$.

$(l_0 - 1)$	n[a]	κ	$\tilde{\tau}_B$[b]	$\tilde{\tau}_S$	$\tilde{\tau}_3$	$\tilde{\tau}_5$
2	0	1.231925	⋯	9.03×10^7	-4.77×10^4	⋯
	1	1.100026	3.32×10^9	1.24×10^8	-3.37×10^4	-3.13×10^{14}
4	0	0.884249	⋯	2.17×10^7	$-\infty$	-5.64×10^9
	1	0.862948	1.93×10^9	4.94×10^7	-1.10×10^7	-1.45×10^{10}
	0	1.643443	⋯	2.13×10^7	$-\infty$	-2.12×10^{15}
	1	1.519573	4.79×10^9	4.77×10^7	-1.92×10^{11}	-2.31×10^{14}

[a]The polytropic index, n, where $p = K\rho^{1+1/n}$. The n=0 case represents the uniform density equilibrium star.

[b]Dissipation due to bulk viscosity is not meaningful for uniform density stars.

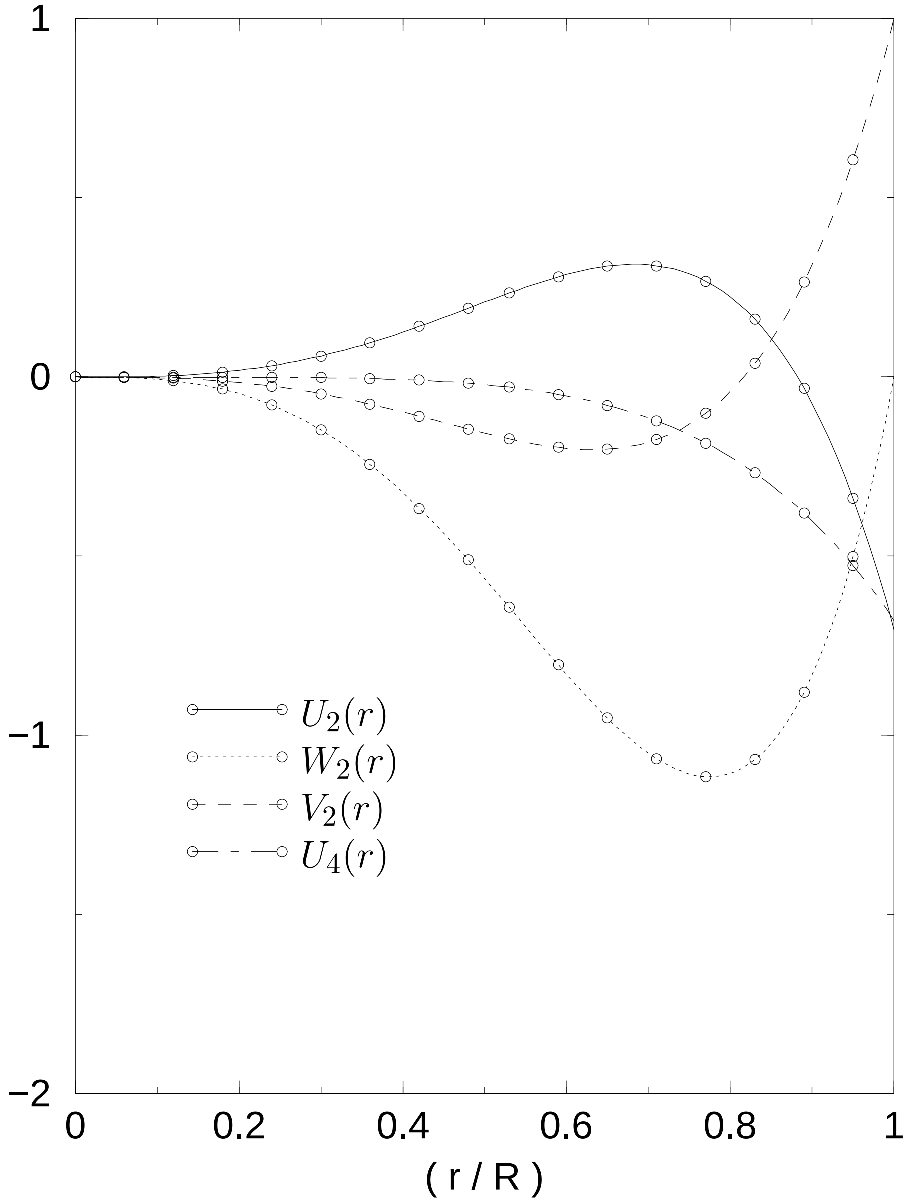

Figure 1

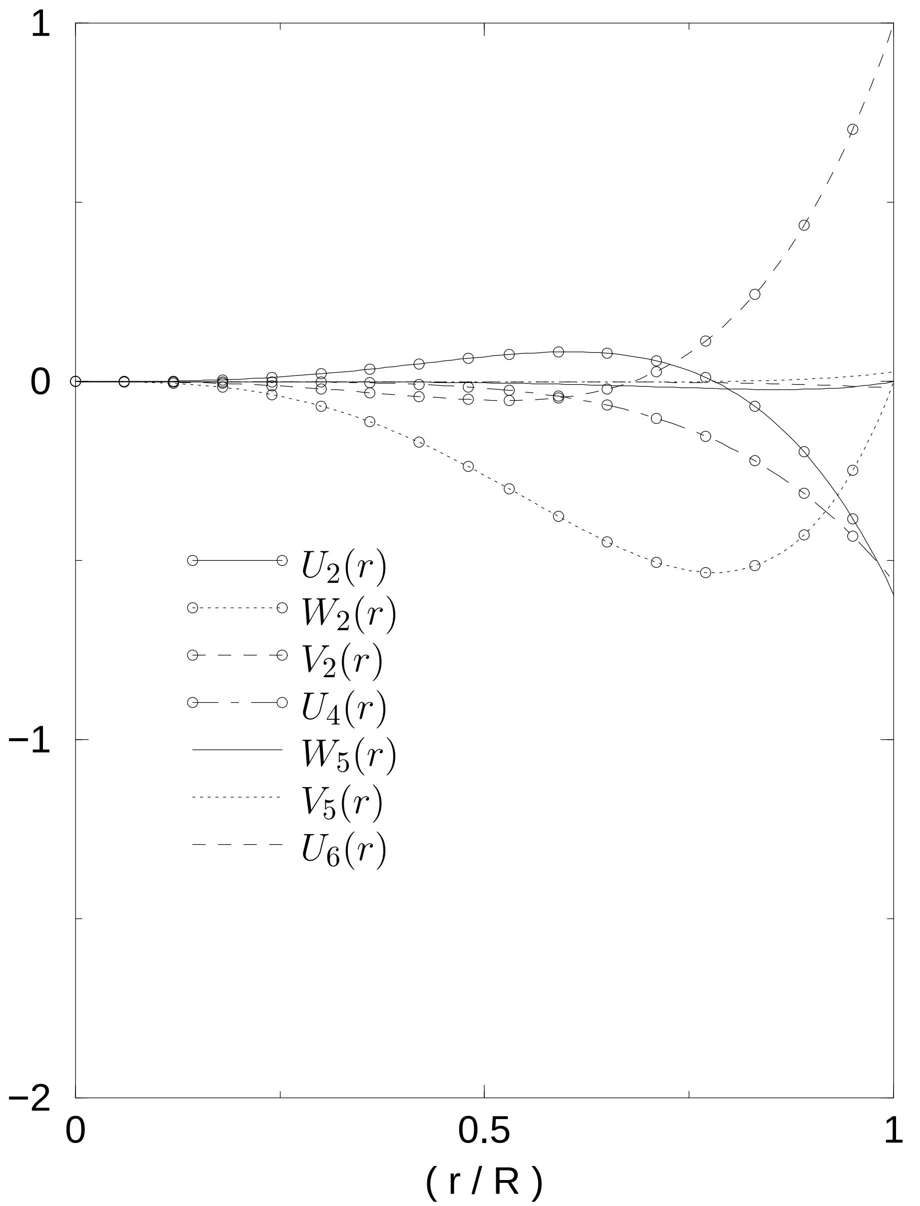

Figure 2

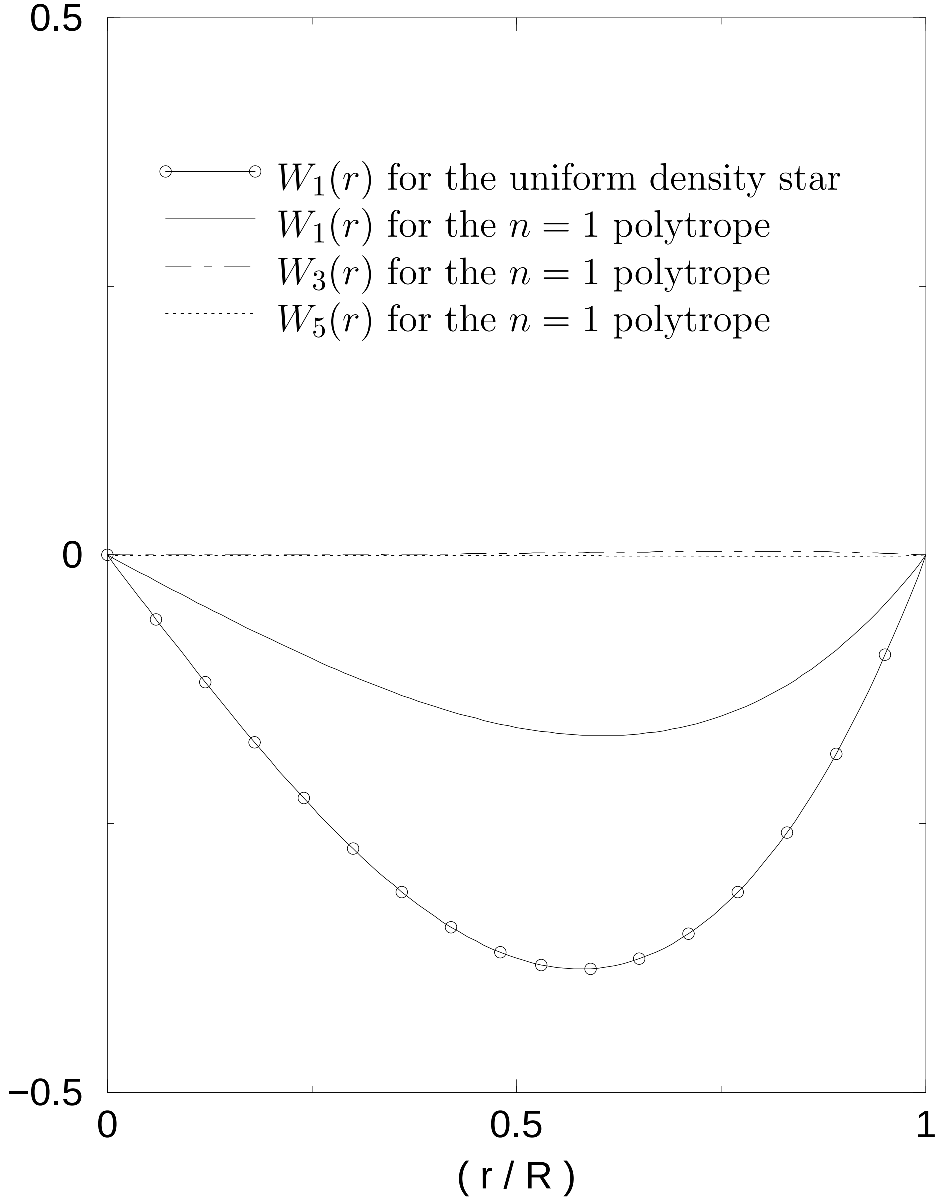
0.5
0
−0.5
0
0.5
1
(r / R)
$W_1(r)$ for the uniform density star
$W_1(r)$ for the $n = 1$ polytrope
$W_3(r)$ for the $n = 1$ polytrope
$W_5(r)$ for the $n = 1$ polytrope

Figure 3

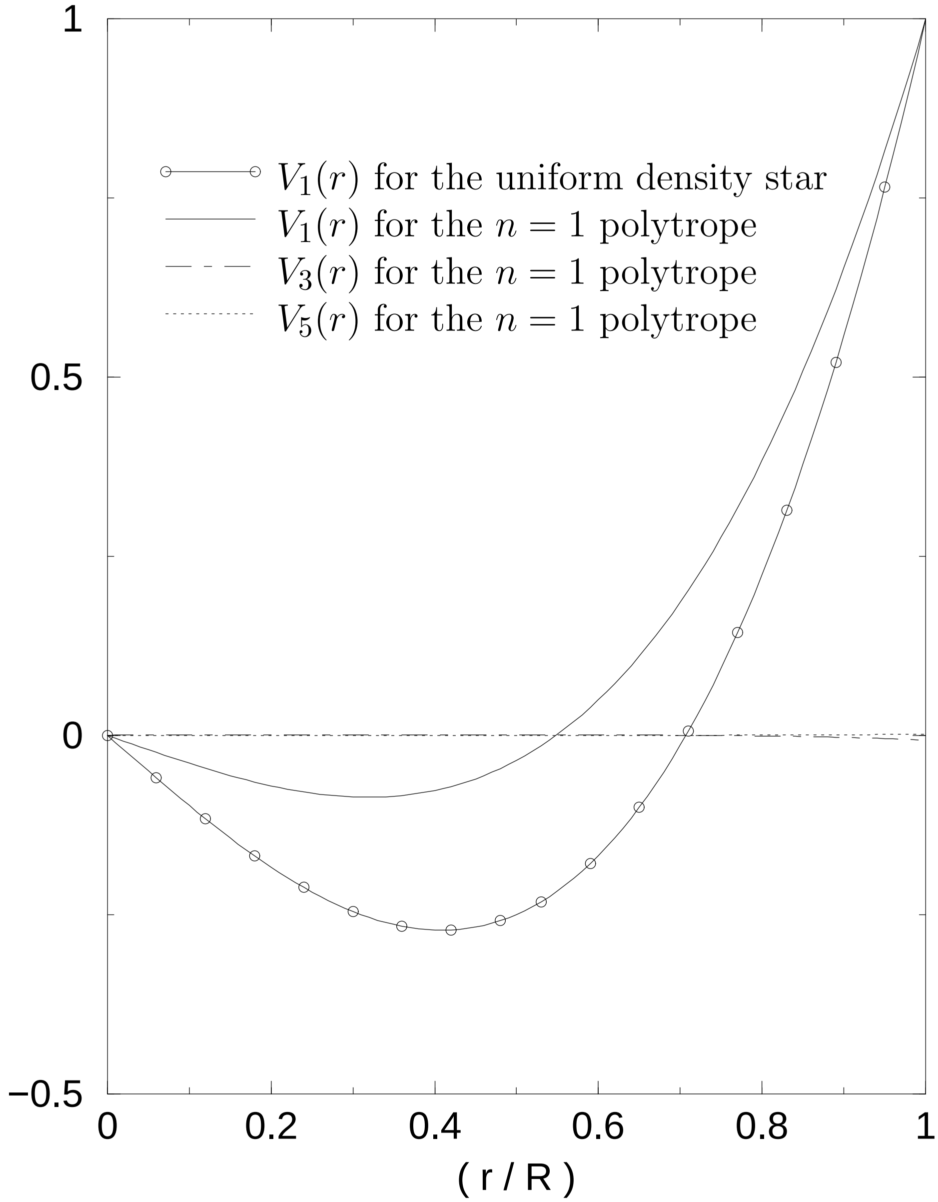
$V_1(r)$ for the uniform density star
$V_1(r)$ for the $n = 1$ polytrope
$V_3(r)$ for the $n = 1$ polytrope
$V_5(r)$ for the $n = 1$ polytrope
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1
(r / R)

Figure 4

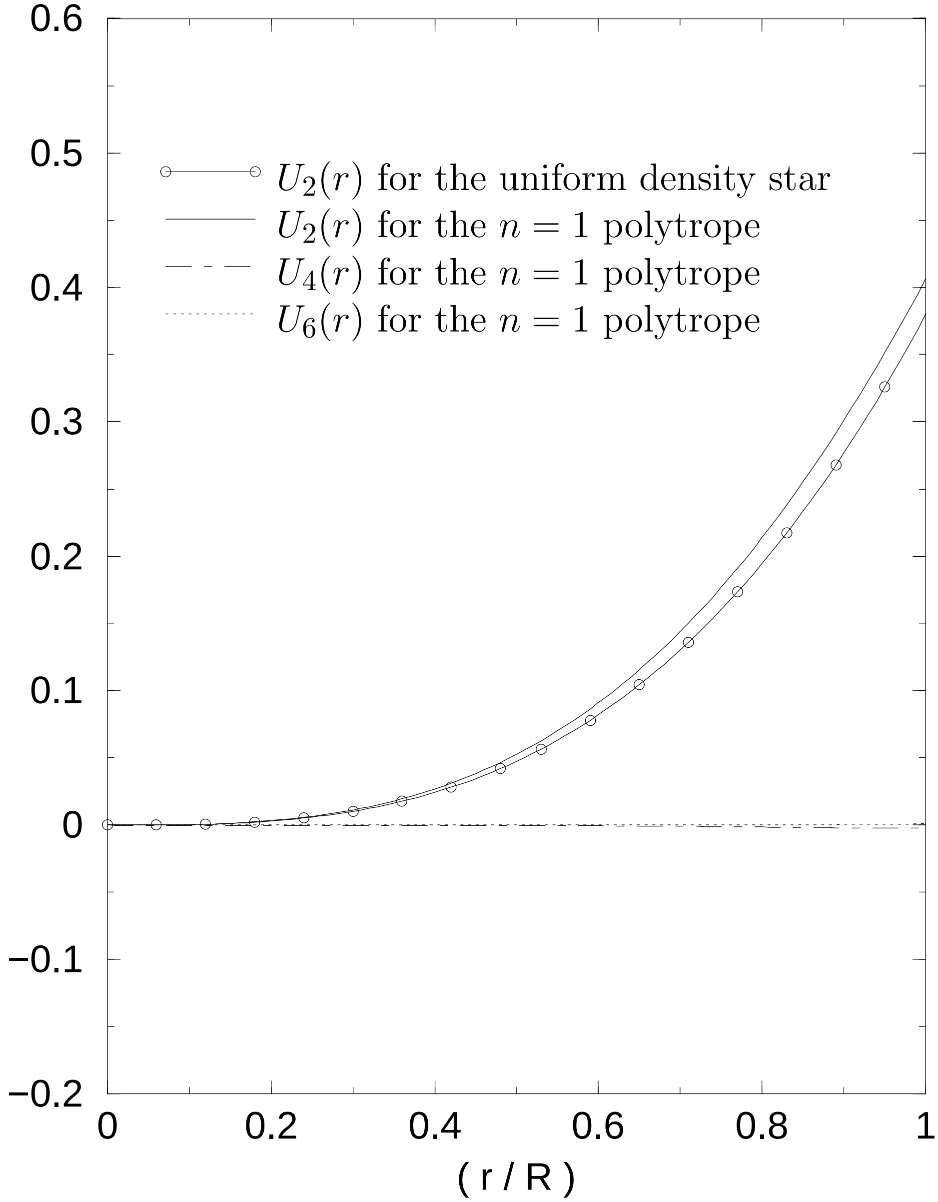
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
$U_2(r)$ for the uniform density star
$U_2(r)$ for the $n = 1$ polytrope
$U_4(r)$ for the $n = 1$ polytrope
$U_6(r)$ for the $n = 1$ polytrope
0
0.2
0.4
0.6
0.8
1
(r / R)

Figure 5

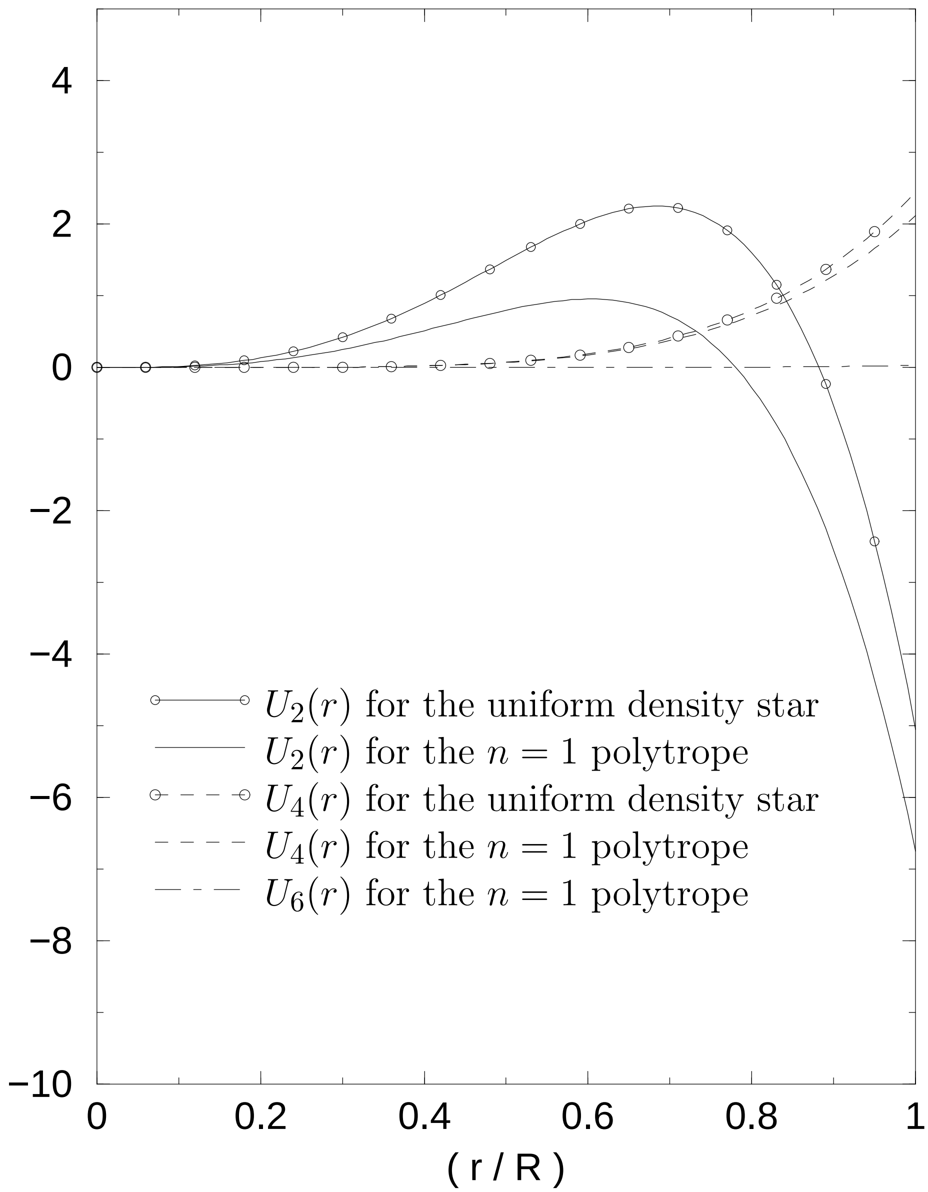
4
2
0
−2
−4
−6
−8
−10
0
0.2
0.4
0.6
0.8
1
(r / R)
$U_2(r)$ for the uniform density star
$U_2(r)$ for the $n = 1$ polytrope
$U_4(r)$ for the uniform density star
$U_4(r)$ for the $n = 1$ polytrope
$U_6(r)$ for the $n = 1$ polytrope

Figure 6

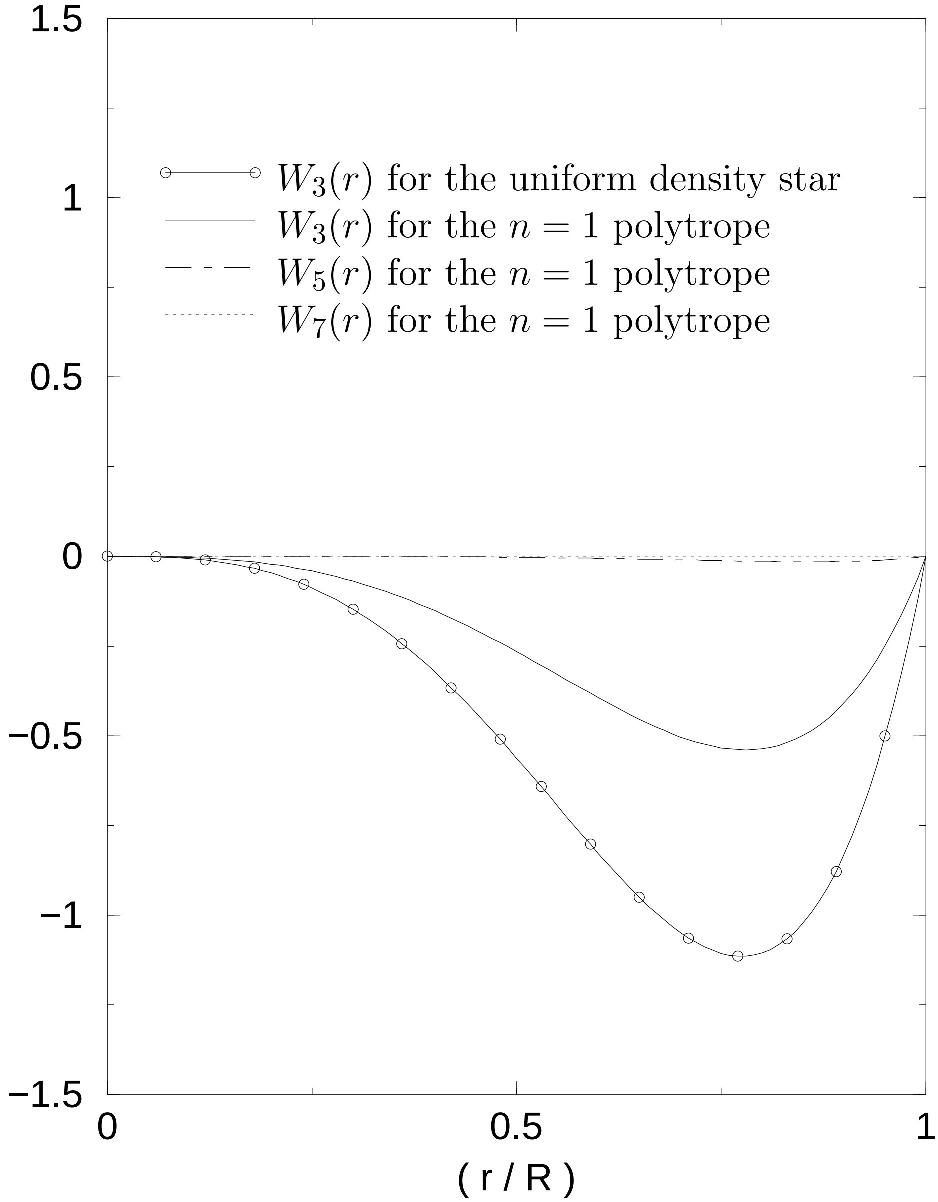
1.5
1
0.5
0
−0.5
−1
−1.5
0
0.5
1
(r / R)
$W_3(r)$ for the uniform density star
$W_3(r)$ for the $n = 1$ polytrope
$W_5(r)$ for the $n = 1$ polytrope
$W_7(r)$ for the $n = 1$ polytrope

Figure 7

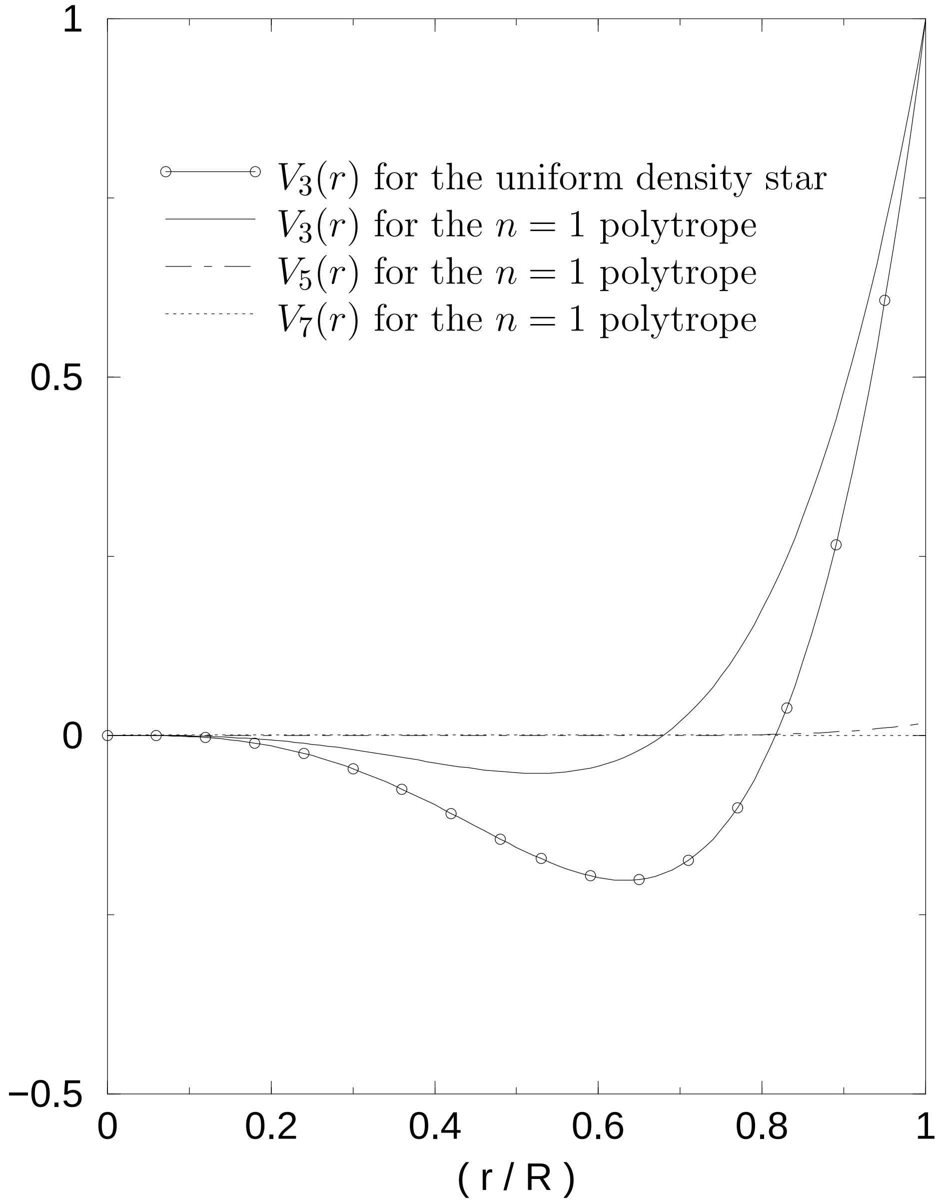
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1
(r / R)
$V_3(r)$ for the uniform density star
$V_3(r)$ for the $n = 1$ polytrope
$V_5(r)$ for the $n = 1$ polytrope
$V_7(r)$ for the $n = 1$ polytrope

Figure 8

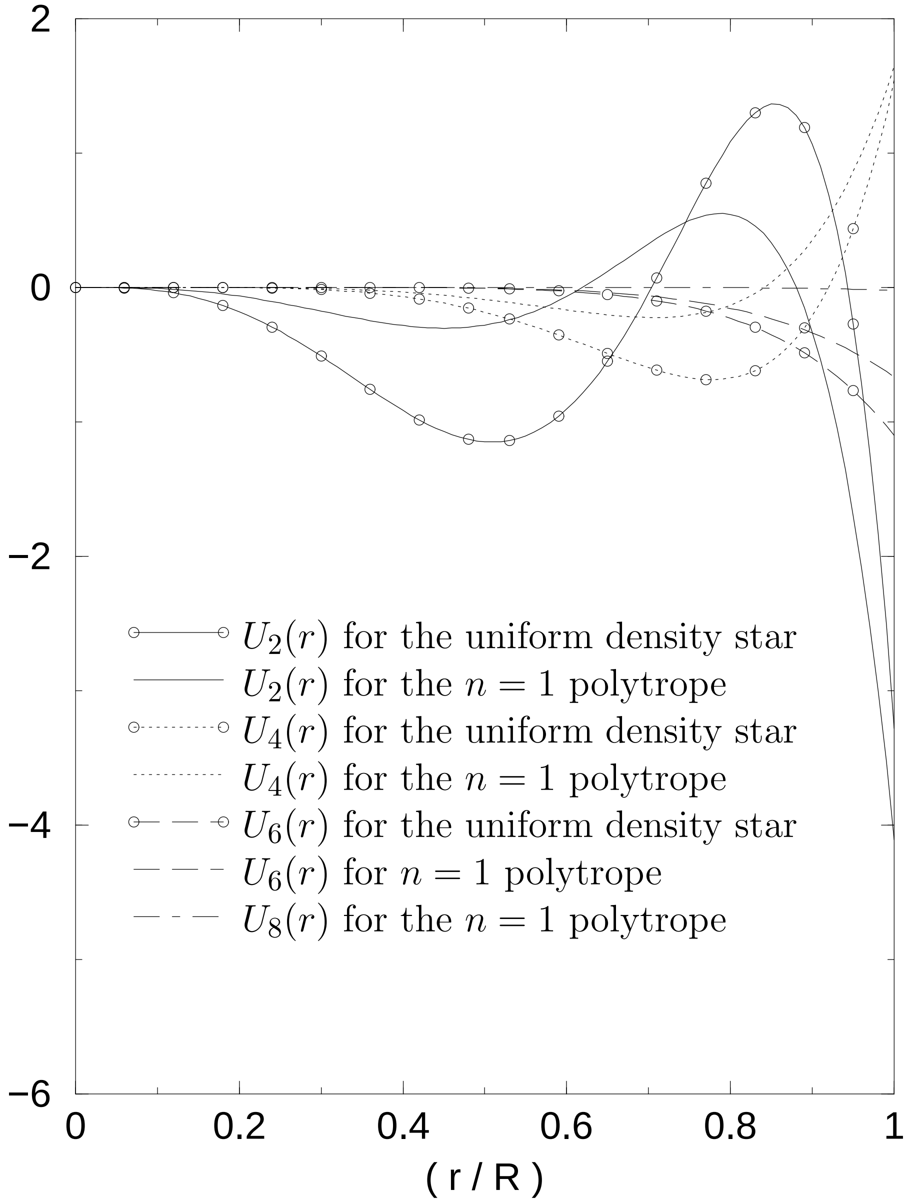
2
0
−2
−4
−6
0
0.2
0.4
0.6
0.8
1
(r / R)
$U_2(r)$ for the uniform density star
$U_2(r)$ for the $n = 1$ polytrope
$U_4(r)$ for the uniform density star
$U_4(r)$ for the $n = 1$ polytrope
$U_6(r)$ for the uniform density star
$U_6(r)$ for $n = 1$ polytrope
$U_8(r)$ for the $n = 1$ polytrope

Figure 9

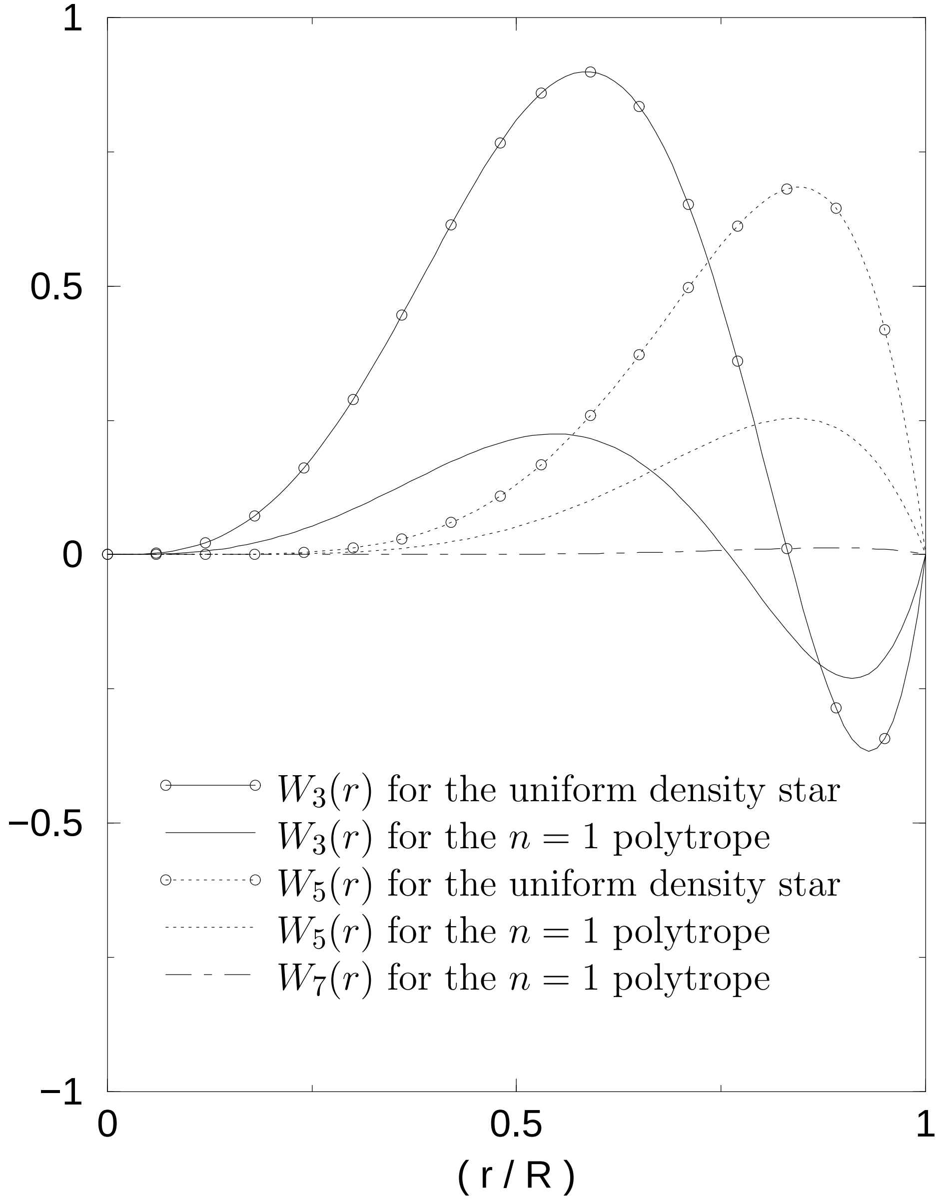
1
0.5
0
−0.5
−1
0
0.5
1
(r / R)
$W_3(r)$ for the uniform density star
$W_3(r)$ for the $n = 1$ polytrope
$W_5(r)$ for the uniform density star
$W_5(r)$ for the $n = 1$ polytrope
$W_7(r)$ for the $n = 1$ polytrope

Figure 10

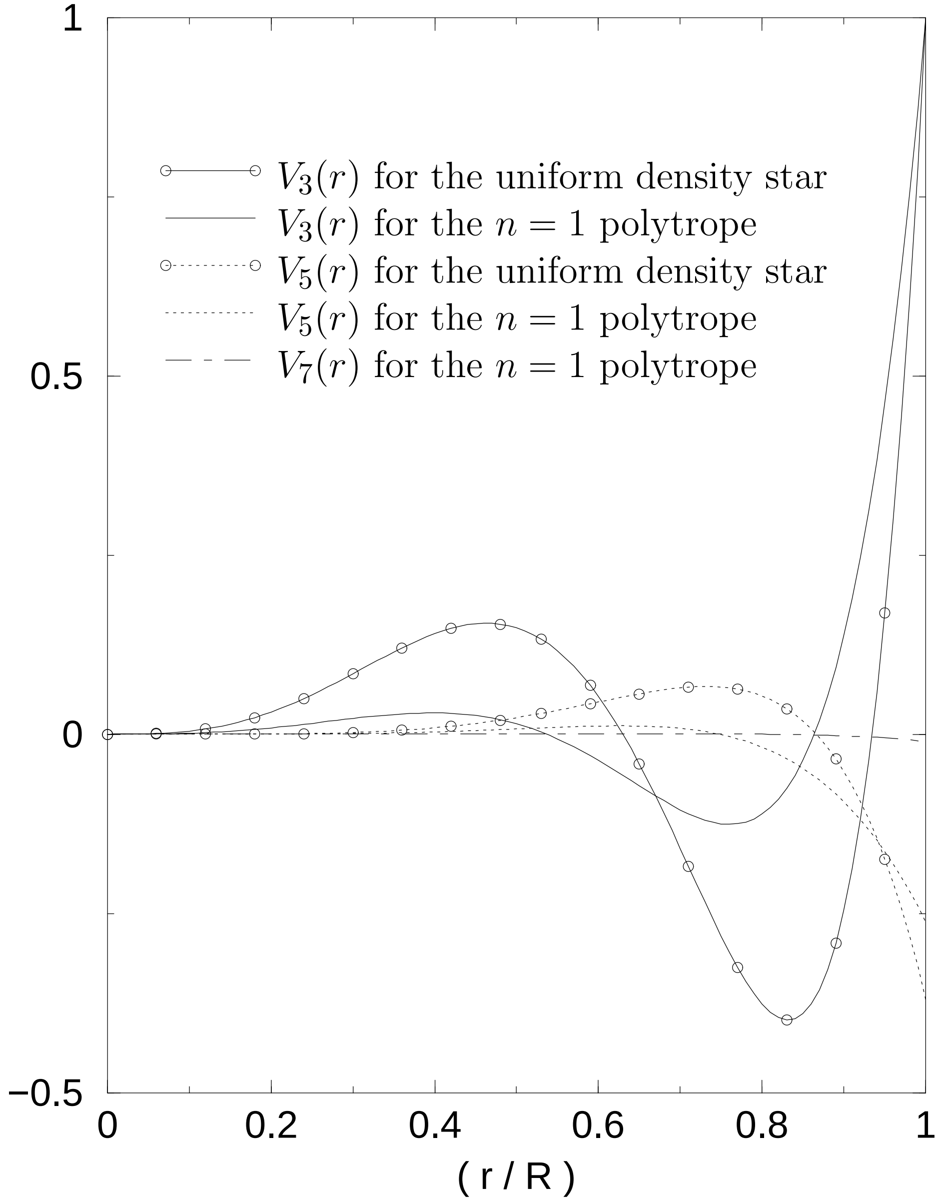
$V_3(r)$ for the uniform density star
$V_3(r)$ for the $n = 1$ polytrope
$V_5(r)$ for the uniform density star
$V_5(r)$ for the $n = 1$ polytrope
$V_7(r)$ for the $n = 1$ polytrope
1
0.5
0
−0.5
0
0.2
0.4
0.6
0.8
1
(r / R)

Figure 11